Chapter 1

Introduction

The exercise and its solution that got me started in writing this book is presented. In later chapters we will occasionally delve deeper into this problem and use aspects of it as an illustration to various areas of mathematics.

1. A drink in a jug has two ingredients, A and B.
   The total volume is one liter. There is as many percent of A as there is permille of B. What are the volumes of A and B?
   Give your answer in decimal form.

For those that needed it, a hint was given to clarify the problem. If the drink contained 90% of A and 90‰ of B then the total would be 90% + 9% = 99%. Trying with 91 would result in a total of 100.1%.

Solution:

```
A:   0.90909090... liter
B:   0.09090909... liter
A+B=0.99999999... = 1
```

Everybody was not convinced with the last equality, which resulted in interesting discussions about the concept of infinity and whether it is possible to regard an infinite number of steps as a completed procedure. We will return to this later. If you believe 1/3 = 0.333… then surely 0.999 ... = 1. The good student would find the solution by setting up an equation.

$$\begin{aligned} x\% + x‰ &= 1 \\ \frac{x}{100} + \frac{x}{1000} &= 1 \quad (\cdot 1000) \\ 10x + x &= 1000 \\ x &= \frac{1000}{11} = 90.9090\ldots \end{aligned} \tag{1.1}$$

How to present fractions in decimal form

Fractions can have a finite number of decimals:

$\frac{1}{2} = 0.5 \quad \frac{7}{4} = 1.75 \quad \frac{17}{8} = 2.125 \quad \frac{37}{125} = 0.296$

If not, they will end with a group of digits that keep repeating.

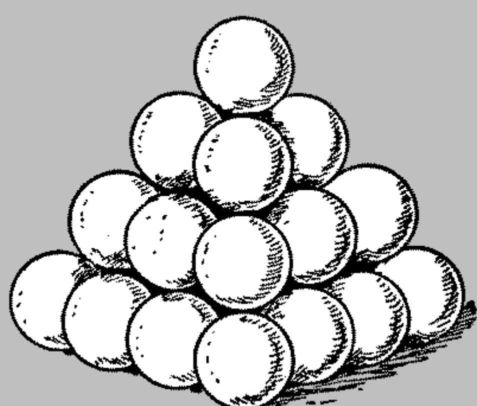
The mathematical symbol for this is a bar above the repeating digits.

$\frac{42}{99} = 0.424242\ldots = 0.\overline{42} \qquad \frac{211}{7} = 30.142857142857\ldots = 30.\overline{142857}$

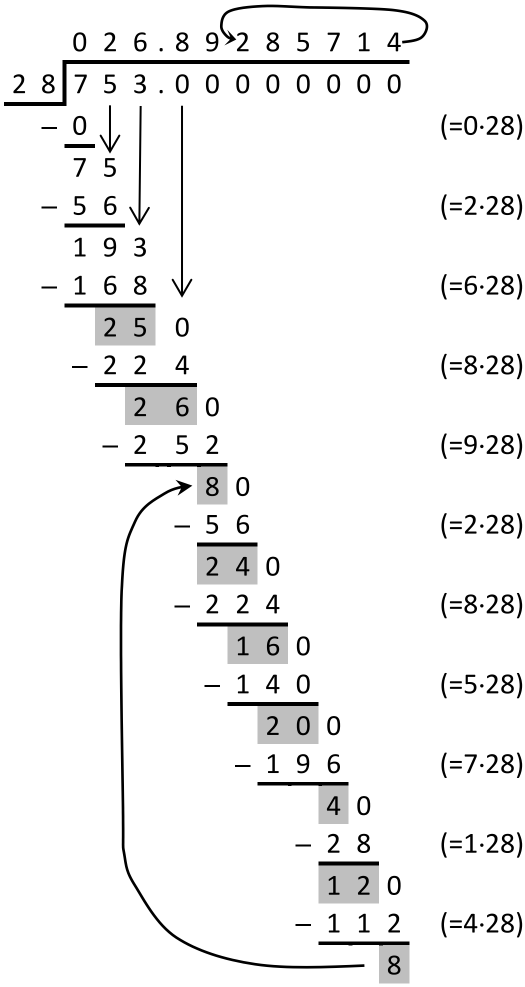
Sphere packing, part 1

A small side remark on the difference between physics and mathematics; in the physical world it turns out that volume is not an additive property. If the liquids are the same it is additive but otherwise it will depend on the shapes and sizes of the molecules and the forces between them. Just think of mixing 5 dl of peas with 5 dl of sand. If you mix 50 ml of water with 50 ml of ethanol you get 96.5 ml of diluted alcohol. Surprisingly there are also examples where the volume will increase. But in the idealized world of mathematics we will not be bothered with such things.

Mathematical research problems are usually hard to understand but not always. A problem related to the example above that is both easy to describe and whose answer seems obvious is the sphere packing problem. If you have a volume and fill it with identical spheres, which arrangement will have the minimal proportion of empty space as the volume goes to infinity? The answer is the regular, layered packaging where each sphere touches twelve other spheres, easy to guess but very hard to prove. Kepler assumed this in 1611 and it became known as the Kepler conjecture. A proof that used a computer program to look for a minimum of a function with 150 variables for each of 5000 different configurations of spheres was presented in 1998 by Thomas Hales. A formal proof that used automated proof checking software was published in 2015. Finding the maximum density with a given proportion of spheres of different sizes as in the pea and sand example would be a hard problem to solve but not necessarily a mathematically interesting problem. To be continued…



To see why digits repeat let us look at the long division of 753 / 28.



Each subtraction in the long division of p/q gives a remainder $r = a - n \cdot q$ $r \in \{0,1,\dots,q-1\}$. At each step a digit from p is added to the remainder. When all the digits in the integer part of p have been handled decimal zeros are used. These remainders are shown in grey. If one of these remainders become zero the division is finished and the number of decimals is finite, otherwise one of the new grey numbers must match one of the previous grey numbers since they come from a finite set. We get a loop of repeating digits.

$$\frac{753}{28} = 26.89\overline{285714} \qquad \textit{general form} \quad \frac{p}{q} = a_1 \dots a_i . b_1 \dots b_j \overline{c_1 \dots c_k}$$

The maximum length of the repeating digits is denominator−1. A fraction with maximal period is $\frac{1}{7}$ with remainder sequence (1,3,2,6,4,5).

Positive numbers are called regular if their decimal expansion is finite, otherwise they are non-regular.

The drink in the first exercise was good, very good but not perfect. It needed a tiny dose of a third ingredient to spice it up and become the most exquisite drink ever made. One day the missing ingredient was found and to get the right taste the proportions of ingredients had to be exactly right.

2. The perfect drink has three ingredients A, B and C.
   There is as many % of A as ‰ of B as ppm of C. (ppm means 10^{-6})
   What proportions of A, B and C are needed to make the perfect drink?
   Give your answer in decimal form.

The solution for the proportion of A is:

$$\begin{array}{l}
0.\overline{90900827197527497500227252067993818743750568130169984} \\
\overline{5468593764203254249613671484410508135624034178711026270} \\
\overline{3390600854467775656758476502136169439141896191255340423} \\
\overline{5978547404781383510589946368511953458776474865921279883} \\
\overline{6469411871648031997091173529679120079992727933824197800} \\
\overline{1999818198345604945004999545495864012362512498863739660} \\
\overline{0309062812471593491500772657031178983728751931642577947} \\
\overline{4593218798291064448686483046995727661121716207617489319} \\
\overline{1528042905190437232978820107262976093082447050268157440} \\
\overline{2327061176256703936005817652940641759840014544132351604} \\
\overline{399600036360330879010999000 0}
\end{array}$$

```
A:      0.90900827197 ... 999000090900827 ...
B:      0.09090082719 ... 099900009090082 ...
C:      0.00009090082 ... 901099900009090 ...
A+B+C=0.99999999999 ... 999999999999999 ... = 1
```

The repeating sequence in A is 576 digits long and this is the only solution. This we know since A = x/100 where x is the unique solution to:

$$\begin{aligned}
x\% + x‰ + x \text{ ppm} &= 1 \\
\frac{x}{100} + \frac{x}{1\,000} + \frac{x}{1\,000\,000} &= 1 \\
10\,000x + 1\,000x + x &= 1\,000\,000 \\
x &= 1\,000\,000/11\,001 = 90.900827\ldots
\end{aligned} \tag{1.2}$$

Making long division by hand to get 576 digits is not recommended. A better way is to turn long division into an algorithm for a computational device. Here are two examples:

TI-83 Calculator

```
Input ”P=”,P
Input ”Q=”,Q
P→R
While R≠0
int(R/Q)→N
R-N*Q→R
10*R→R
Disp N
Pause
End
```

C programming language

```
#include <stdio.h>

main()
  {
  int dividend,divisor,
      quotient,remainder,
      decimals,counter;

  printf("Dividend: ");
  scanf("%d",&dividend);
  printf("Divisor: ");
  scanf("%d",&divisor);
  printf("Max number of decimals: ");
  scanf("%d",&decimals);
  remainder=dividend;
  counter=0;
  while(remainder!=0 && counter<=decimals)
    {
    quotient=remainder/divisor;
    remainder=(remainder-quotient*divisor)*10;
    printf("%d",quotient);
    if (counter==0) printf(",");
    counter++;
    }
  }
```

Programming is a very useful tool to turn creative ideas into reality. A quicker way to explore mathematics is to use a program like Geogebra or Mathematica. I will use the latter in this book. A central part of Mathematica notation is lists. A Rectangular array of numbers is handled as a list of lists. Exploring the decimal expansion of 753/28 in Mathematica would look like:

```
In[1]:= RealDigits[753/28]
Out[1]= {{2,6,8,9,{2,8,5,7,1,4}},2}
```

It is a list starting with the digits that precede the repeating digits and ending with a new list containing the digits that repeat. The last number in the original list gives the position of the decimal point. To extract element 4 from a list you would put [[4]] after the list. To count from the end of the list, use a negative number. [[m]][[n]] can be written as [[m , n]]. The number of elements in a list is given by Length[*list*].

To get the number of repeating digits in $\frac{1\,000\,000}{11\,001}$ you could write:

```
In[2]:= Length[RealDigits[1/11001][[1,-1]]]
Out[2]= 576
```

In the preface I mentioned a problem with an unexpected and impressive answer. This is the exercise:

3. $A + B + C = 1 \quad B\ is\ 1\%\ of\ A\ and\ C\ is\ 1$ ‰ of B.
   Find the missing digits x, y, z, … in the following system:

```
A:     0.xyz........
B:     0.00xyz......
C:     0.00000xyz...
A+B+C=0.99999999...
```

Solving the equation as before, $A = \frac{100\,000}{101\,001} = 0.\overline{990089207 \ldots 099900000}$, 16 640 digits that repeat over and over. To print them you need six A4-pages.

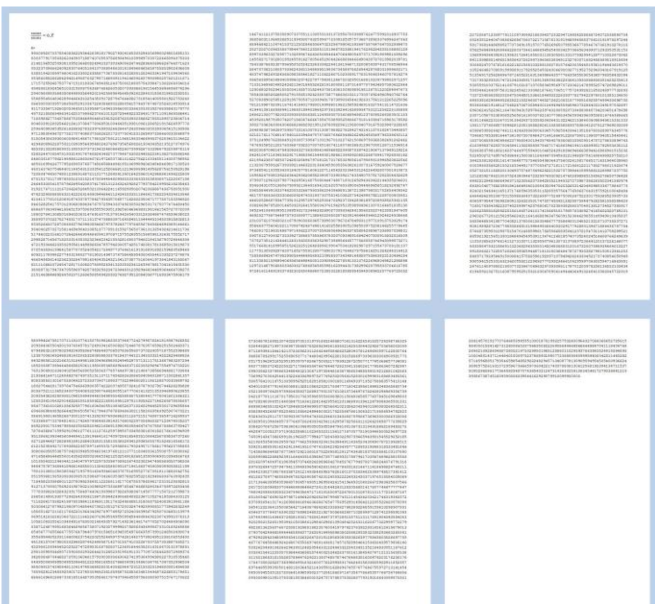


Fig. 1a Periodic part of the solution to exercise three.

Setting $A + B + C + D = 1$ where B=1% of A, C=1 ppm of B, D=1‰ of C results in numbers with 25 014 018 913 repeating digits which would make a row with 4 mm per digit 100 000 km long. The earth's equator is 40 000 km.