Chapter 2

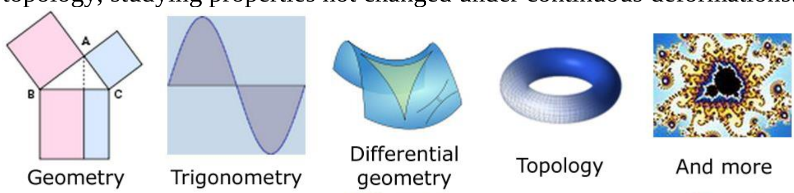
Origins

The theme of this chapter is the history of mathematics. We try to find answers to what mathematics is, its historical roots and main branches.

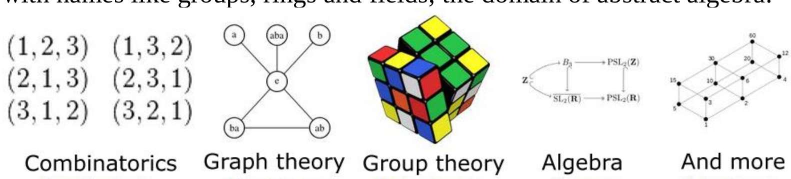
2.1 What Is Mathematics?

Mathematics is sometimes described as the study of quantity, space, structure and change. These fields are called **pure mathematics**.

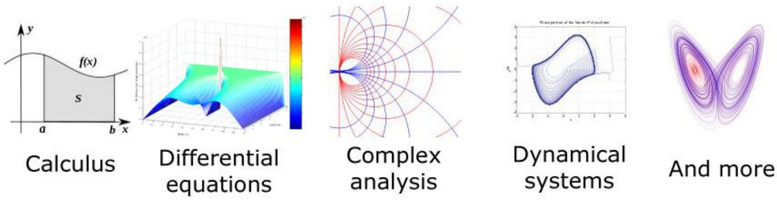
- **Quantity** would mean the study of different number systems, doing calculations with them (arithmetic) and studying their properties.
  - Natural numbers: $\mathbb{N} = \{1 , 2 , 3 , \ldots\}$
  - Integers: $\mathbb{Z} = \{0, \ \pm 1 , \ \pm 2, \ldots\}$
  - Rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z} \ and \ q \neq 0\}$
  - Real numbers $\mathbb{R}$ which includes $\mathbb{Q}$ and all its limits, such as $\sqrt{2}$ and π.
  - Complex numbers: $\mathbb{C} = \{a + b \cdot i |\ a, b \in \mathbb{R}\}$
  - And more
- **Space** would start from plane geometry and grow to include fields like trigonometry, geometry of curved objects in various dimensions and topology, studying properties not changed under continuous deformations.



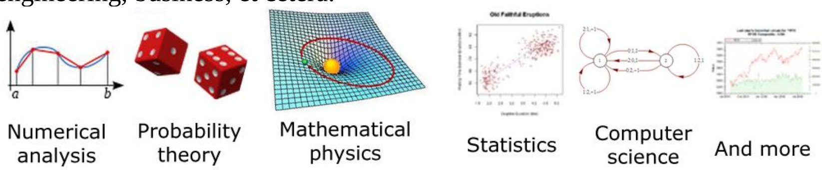
- **Structure** would study discrete systems like combinatorics and graph theory, unify similar mathematical structures into axiomatic frameworks with names like groups, rings and fields; the domain of abstract algebra.



- **Change** would study dependencies defined by functions. This includes calculus with derivation, integration and differential equations. Quantities involved can be real, complex or more structured.



In addition to pure mathematics there is **applied mathematics** with stronger links to practical problems or special applications in natural or social science, engineering, business, et cetera.



Each subfield can be described well enough. Two problems with this explanation is the fuzziness of “and more” and mathematics is more unified than suggested. New fields arise and different fields come together as mathematics develops. When coordinate systems were introduced geometric and algebraic methods became less separated.

In mathematics it is possible to separate correct answers from wrong answers, but the division of mathematics itself has no correct answer. Another way to illustrate the mathematical landscape could look like this:

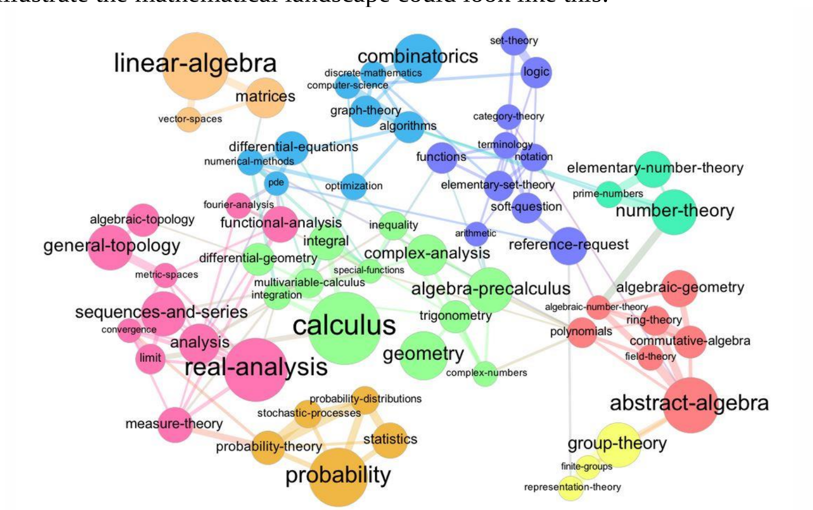


Fig. 2.1.1 Mathematical landscape.

There is an official classification scheme used for research papers. It is the Mathematics Subject Classification (**MSC**) which has three levels, a two digit number, a letter and another two digit number. A paper classified with 53A45 would be about differential geometry - classical - vector and tensor analysis.

The top level contains 63 mathematical disciplines. The grouping in the list below is informal and not part of MSC.

General and foundations of mathematics and logic, 00-03

00: A general area including recreational and philosophy of mathematics.
01: History and biography
03: Mathematical logic and foundations with subjects like set theory, model theory, proof theory and computability theory

Algebra and discrete mathematics, 05-22

05: Combinatorics and graph theory
06: Order theory; lattices, ordered algebraic structures
08: Algebraic systems
11: Number theory
12: Field theory and polynomials
13: Commutative rings and algebras
14: Algebraic geometry
15: Linear and multilinear algebra
16: Associative rings and associative algebras
17: Non-associative rings and non-associative algebras
18: Category theory and homological algebra
19: K-theory
20: Group theory
22: Topological groups and Lie groups

Analysis, 26-49

26: Real functions
28: Measure and integration
30: Complex functions
31: Potential theory
32: Several complex variables and analytic spaces
33: Special functions
34: Ordinary differential equations
35: Partial differential functions
37: Dynamical systems and ergodic theory
39: Difference equations and functional equations
40: Sequences, series and summability
41: Approximations and expansions

42: Harmonic analysis, including Fourier analysis
43: Abstract harmonic analysis
44: Integral transforms, operational calculus
45: Integral equations
46: Functional analysis
47: Operator theory
49: Calculus of variation and optimization

Geometry and topology, 51-58

51: Geometry
52: Convex geometry and discrete geometry
53: Differential geometry
54: General topology
55: Algebraic topology
57: Manifolds
58: Global analysis, analysis on manifolds

Applied mathematics, 60-97

60: Probability theory, stochastic processes
62: Statistics
65: Numerical analysis
68: Computer science
70: Mechanics
74: Mechanics of deformable bodies
76: Fluid mechanics
78: Optics, electromagnetic theory
80: Classical thermodynamics, heat transfer
81: Quantum theory
82: Statistical mechanics, structure of matter
83: Relativity and gravitational theory
85: Astronomy and astrophysics
86: Geophysics
90: Operations research, mathematical programming
91: Game theory, economics, social and behavioral sciences
92: Biology and other natural sciences
93: Systems theory, control theory
94: Information and communications, circuits
97: Mathematics education

There is no agreed upon definition to what mathematics is so let us conclude with a definition from the Columbia Encyclopedia (5th ed.) and then some personal answers from various mathematicians.

Mathematics: deductive study of numbers, geometry, and various abstract constructs, or structures. The latter often arise from analytical models in the empirical sciences, but may emerge from purely mathematical considerations.

- Mathematics is that which mathematicians do.
- Mathematics is the study of well-defined things.
- Math is the study of statements of the form $P \Rightarrow Q$ (Bertrand Russel).
- Mathematics is the science of patterns (Keith Devlin)
- In mathematics the art of asking questions is more valuable than solving problems (Georg Cantor)
- God made the integers, all else is the work of man. (Leopold Kronecker) *Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*
- Mathematics is the language with which God has written the universe. (Translated version from Galileo Galilei's book *Il Saggiatore* 1623)
- The essence of mathematics is in its freedom (Georg Cantor)

2.2 Etymology

Etymology, the history of words often reflects the history of the concepts that the words describe. Mathematics comes from the Greek μάθημα (máthēma). In ancient Greek it meant ”what one learns”. Its modern meaning is lesson and máthēma comes from μανθάνω (manthano) which means “to learn”. Advanced mathematics was done in other places long before its golden age in Greece but it is still a telling origin since mathematics as we use the word today with axioms and formal proofs dates back to Thales who lived around 600 BC, the first of the great Greek thinkers.

The use of the word “mathematics” can be traced to the Pythagoreans. Pythagoras himself (570−495 BC) has become a mythical figure in mathe– matics but was probably more involved with religious questions about the soul, rituals and promoting a strict diet and lifestyle. He grew up on the island of Samos near the coast of modern Turkey where he would be influenced by Thales and his pupil Anaximander. He made extensive travels to study before he settled in the Greek colony Croton (southern Italy) where he established a school of followers. Many of the accomplishments ascribed to Pythagoras were probably made by his successors. The Pythagoreans formed closed and secretive communities. Their motto was “all is numbers” and their beliefs combined mathematics, religion and music. The school would later split into two groups, akousmatikoi, (“listeners“) that focused on religious, mystical and ritualistic aspects and mathēmatikoi (“teachers”).

The meaning of “mathematics” has shifted over time. Before the 18th century it often meant astrology or sometimes astronomy, their difference was not so obvious in those times. When Saint Augustine (354-430) warned people of mathematici he meant astrologers, not mathematicians. The transition to the modern use of mathematics was a gradual process over several centuries.

A more recent precursor of “mathematics” is the Latin word “mathematica”. It is a word of neutral grammatical gender in plural form, based on the Greek plural τα μαθηματικά (ta mathēmatiká) meaning “all things mathematical”. This explains the plural form of mathematics and the same goes for physics. The French use plural from for mathematics but singular form for physics, “mathématiques” and “physique”, just as Britons and Americans use different short forms “math” and “maths”. Guess who uses the more traditional form.

2.3 Prehistoric Mathematics

The origins of mathematics must have been preceded by some kind of cognitive development and ability for abstractions. In the evolution of different life forms it appears that this has happened several times. Basic problem solving skills can be seen in various groups of animals. Not only chimpanzees and dolphins, also corvids and parrots are studied for their cognitive abilities.[1,2] Even among invertebrates with radically different nervous systems there are “smart” ones. Cephalopods are considered the most intelligent and octopuses can figure out how to open a screw cap to get dinner inside a jar.[3]

The first step in a mathematical career would be to count. Research on rhesus monkeys indicates that their conceptual understanding of numbers is limited to very small numbers.[4] There is no solid evidence for counting among our

Hominid ancestors. Maybe there is simply no need for counting in a hunter-gatherer society. The Guiness Book of World Records says that the word for "three" among the Yanco tribe in the Amazon is "poettarrarorincoaroac" and they do not have any words for bigger numbers.

Fig. 2.3.1 Ishango bone, top and wolf bone, bottom.

Many texts on the history of mathematics mention the Ishango bone as the earliest known artefact with mathematical meaning. It was discovered in 1960 in the Belgian Congo and is dated to 20,000 BC. An even older bone, believed by some to indicate counting is a wolf bone found in 1937 in Czechoslovakia. It is dated to be 30,000 years old. These claims do not look very convincing to me and there are plenty of alternative explanations for the markings on the bones.[5]

It might seem strange that people with the same cognitive abilities as us, like the artists behind the Lascaux cave paintings 17,000 years ago, might not have had access to the concept of numbers, but history teaches us that concepts that now seem obvious like zero and negative numbers took several centuries to become generally accepted after their discovery and were discarded by many of the best mathematicians of the time.

Solid evidence of numeracy and its origin can be found in artefacts from the Middle East. Archeologist Denise Schmandt-Besserat searched museums for the oldest human use of clay and found thousands of small clay objects, some as old as 10,000 years. These clay tokens were used for bookkeeping and each token represented a different object, such as a jar of oil, a skin or an animal. There was one token for one sheep and a different token for 10 sheep. Tokens often had holes in them so that they could be used like beads on a bracelet or necklace. These figurines were possibly the first use of a currency.

Fig. 2.3.2 Clay tokens from Susa in present-day Iran.

Towards 6000-5500 BC there are finds of sealed hollow clay balls containing these coins. Many clay balls had description of their content on the outside. Later the balls were replaced by clay tablets with cuneiform writing. The French archeologist argues that the origin of writing in the Sumerian culture in Mesopotamia around 4000 BC and the first currency came from these bookkeeping devices, used for counting.[6]

The introduction of abstract numbers came in stages. At first there was an ovoid token to represent a jar of oil. By 3100 BC it had been replaced by two signs on a clay tablet, one sign for the numeral and one sign for a full jar of oil. Later a special sign was introduced for oil; in all three signs, a numeral, a unit and a commodity. The numerals 10, 60, 100, 600, 3600 and 36000 had special tokens and later signs. The use of sixty is no coincidence. It has unique properties, $60 = 2^2 \cdot 3 \cdot 5$ making it divisible by 2, 3, 4, 5, 6 and 10. Our division of hours and minutes into 60 units, and a turn into 360 degrees can be traced back to these tokens from 3000 BC.

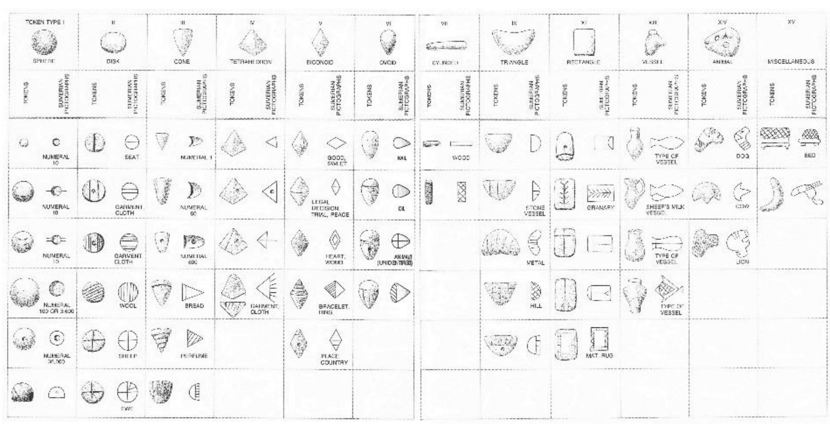


Fig 2.3.3 Tokens and pictographs in the earliest Sumerian inscriptions.

Anatomically modern humans (AMH) are about 200,000 years old and their cognitive abilities must have been comparable to ours for much more than 5,000 years. Why did it take so long for them/us to dicover how to count?

A small group of hunter-gatherers could do well without counting very much but a society based on agriculture and trade would have a much bigger need for basic mathematics. So why did agriculture not come sooner?

The introduction of agriculture coincides with the start of the Holocene 11,700 BP. This geological epoch is an interglacial that was initiated by a climate change that resulted in an unusually long period of warm and stable climate. The previous period, Late Pleistocene (126,000-11,700 BP) was dominated by glaciation and climate variation unfavourable for prolonged periods of agriculture that could result in a more complex society. Pleistocene and Holocene are the two periods of the present ice age that started 2.6 million years ago and which might come to an end as we now enter the Anthropocene and a very unpredictable future.

2.4 Babylonian Mathematics

Babylonian mathematics stretches from the days of the early Sumerians to 539 BC when Cyrus, king of Persia conquered Babylon. It encompasses different peoples, cultures, empires and languages but a common factor is the writing and number systems with their roots in Sumerian clay tokens. Its geographical area was Mesopotamia, the fertile plain between the Euphrates and Tigris rivers. For most of its time the center was Babylon a city dating back to 2300 BC and the Akkadian Empire.

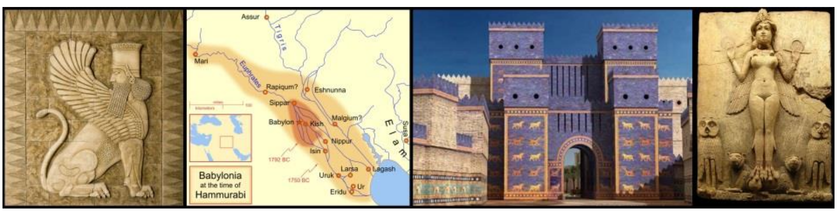


Fig. 2.4.1 Mythological Lamassu, Mesopotamia, Ishtar gate to Babylon, Ishtar.

By 2000 BC the old Sumerian system with separate symbols for 600, 3600 and 36000 was replaced with a positional number system consistently based on positive and negative powers of 60, a sexagesimal system. Sexagesimus is latin for 60th and Sexagesima is the second Sunday before Ash Wednesday, the first day of Lent 46 days before Easter, in total about 60 days.

The sexagesimal system of the old Babylonian period needed 59 digits since they did not have a digit for zero. Instead of using 59 different symbols they used a Roman type system with symbols for 1 and 10.

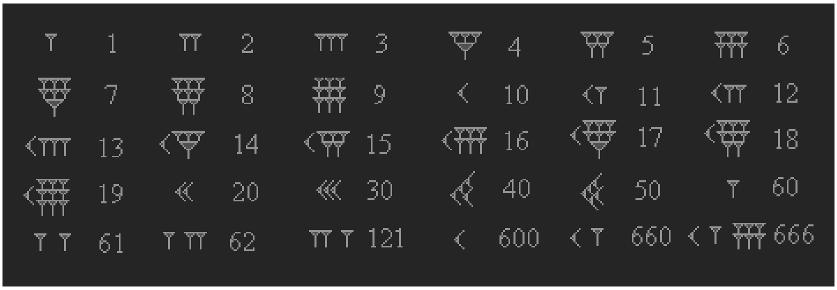


Fig. 2.4.2 Babylonian numerals.

Even after separating digits with an extra space there is still ambiguities due to the lack of zero and decimal point. 𒁹 𒁹ld mean both 61, 3601 and $1 + 1/60$. It could be any number $60^m + 60^n$ with $m \neq n$ and $m, n \in \mathbb{Z}$, context would decide. A placeholder symbol playing the role of zero was introduced during Nebuchadnezzar's rule.

Much of our knowledge of Babylonian mathematics comes from some 400 clay tablets with cuneiform script, dating from 1800-1600 BC. The soft clay was inscribed with a wedge-shaped stylus before being baked. The Latin word for "wedge" is *cuneus.*

Tablets from 2000 BC give lists of squares for numbers up to 59 and cubes for numbers up to 32. Like logarithmic tables they were used to do difficult multiplications without having to multiply: $ab = [(a+b)^2 - (a-b)^2]/4$. Evidently basic school algebra has been around for 4 000 years. They did of course not use our notation but the basic ideas were known. Not only could they do algebra, they also solved equations of type $x^2 + bx = c$ and even some cubic equations.

$$ax^3 + bx^2 = c \quad \left(\cdot \frac{a^2}{b^3}\right) \quad \Rightarrow \quad \left(\frac{ax}{b}\right)^3 + \left(\frac{ax}{b}\right)^2 = \frac{ca^2}{b} \quad \Rightarrow \; y^3 + y^2 = d$$

Approximate solutions to cubic equations can be found by using a table with columns for n and $n^3 + n^2$. Such tables have been found.

Divisions were done by multiplication with reciprocals $x \cdot (1/y)$. A bit odd is that such tables only have reciprocals for integers with finite sexagesimal expansion. Did they ever know that 1/59 would be 𒁹𒁹𒁹…?

Geometry has a natural place in any society involved with architecture and agriculture. The Babylonians knew of Pythagoras' theorem a thousand years before Pythagoras lived. One of the most famous clay tablets is called Plimpton 322. It was written about 1800 BC and it contains list of fifteen Pythagorean triples. A Pythagorean triple is three integers (a, b, c) that satisfy $a^2 + b^2 = c^2$. It is not known for cerrtain if they knew that all Pythagorean triples can be derived from $(p^2 - q^2, 2pq, p^2 + q^2)$.

YBC 7289 is another interesting clay tablet from 1800 – 1600 BC. It belongs to the Yale Babylonian Collection and it shows a square with two diagonals. By the side you can see 30 or ½ written in cuneiform script and on the diagonal are two numbers, (1;24;51;10) and (42;25;35).

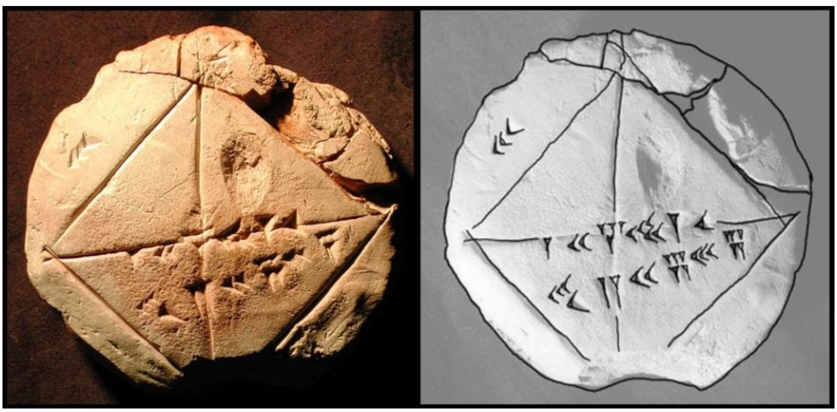


Fig. 2.4.3 Clay tablet YBC 7289.

The diagonal of a unit square is $\sqrt{1^2 + 1^2} \approx 1.4142136$ and the diagonal of a square with side ½ is $\sqrt{(1/2)^2 + (1/2)^2} \approx 0.7071068$.

$$(1\,.\,24\ 51\ 10\,)_{60} \approx 1.4142130_{10} \quad (0\,.\,42\ 25\ 35\,)_{60} \approx 0.7071065_{10}$$

How could they find such good approximations to square roots? A possible answer is hinted in exercise 2.2.

Despite the many changes of rulers in Mesopotamia, there was a continuity in mathematics from ancient times to the time of Alexander the Great 330 BC.

Historical dates of Near Eastern Bronze and Early Iron Age vary by more than a century depending on chronology. Middle chronology is used for old dates in the following table of rulers in Mesopotamia.

Time (BC)	Era / Language	Center	Note
3500 – 2330	Sumer Sumerian	Ur, Nippur	Gilgamesh 2650±150
2330 – 2150	Akkadian Empire Akkadian	Akkad	Sargon –2280
2110 – 2000	Neo-Sumerian Empire Sumerian	Ur	Ended by Elamites
2030 – 1740	Early Amorite city-states Akkadian	Larsa Babylon	
1890 – 1590	1st dynasty of Babylonia Akkadian	Babylon	Hammurabi –1750 Venus tablet ~1640 Ended by Hittites
1600 – 1220	Kassite dynasty Kassite, Akkadian	Dur-Kuigalzu	
1220 – 730	Babylonian dynasties (Chaldean, Elamites, etc.)	Babylon	
910 – 620	Neo-Assyrian Empire Akkadian, Aramaic	Aŝŝur Nineveh	Ashurbanipal –630
620 – 540	Neo-Babylonian Empire Akkadian, Aramaic	Babylon	Nebuchadnezzar –560
615 – 550	Median Empire Median (Iranian)	Hamadan	Conquered by Cyrus
540 – 330	First Persian Empire Median, Akkadian, Persian, Greek etc	Babylon	Cyrus II –530 Darius I –486 Xerxes I –465
330 – 140	Seleucid Empire Greek, Persian, Aramaic	Seleucia Antioch	Alexander III –320 Seleucus –280

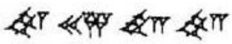
Timetable for Mesopotamia from 3500 BC to 140 BC.

There is an unbroken link from our use of 60 in units of time and 360 in units of angles. The Sumerians and early Babylonians did not use angles until much later when they became the first to sytematically record positions of stars and planets on the celestial sphere. Their knowledge was picked up by Greek astronomers and through the Romans the use became widespread in European languages. The Latin phrase for the digit in position 1/60 of an angle was “pars prima minuta” (fist small part) and for the digit in position 1/3600 “pars secunda minuta” (second small part). Hence our names for minute and second. Sexagesimal ‘decimals’ were used to express small numbers in Europe from the Greeks and onwards. The decimal system for integers was introduced in 1202 by Leonardo of Pisa in “Liber Abaci” (*The book of calculation*). He is more known under his nickname Fibonacci.

Decimal representation of fractions didn't take off until 1585 when Simon Stevin from Flanders wrote a 35-page book "De Thiende" (*The art of tenths*).

Astronomy and navigation still use
Sexagesimal notation for angles
55°34'31" N 12°49'36" E

° degree: 1 turn = 360° = 1τ
′ arcminute: $1° = 60'$
″ arcsecond: $1' = 60''$
$\tau/7 = 51.\overline{428571}° =$
$51°\,25'\,\overline{42.8571}'' \approx$



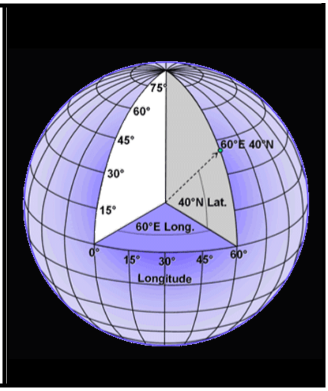


Fig 2.4.4 Contemporary sexagesimal notation.

Babylonian mathematics was not an investigation of an independent reality. It was a tool for practical applications. Negative and irrational numbers were not needed and unknown. Algebraic procedures were described verbally and justified empirically without formal proofs. They may or may not have known of periodic fractions but they never knew that $\sqrt{2}$ would need an infinite string of digits. That proof would have to wait for a thousand years after the making of YBC 8279.

2.5 Egyptian Mathematics

Fig 2.5.1 Pyramids of Giza, Tutankhamun and papyrus art.

Mathematics in the early Egyptian civilization developed independently of the contemporary Sumerian culture. The two kingdoms of Egypt were unified some time between 3500 and 3000 BC. The oldest pyramid, the step pyramid of Djoser was built (2630–2610 BC) in the third dynasty of the Old Kingdom. The architect and high priest Imhotep has become a legendary figure, dubbed both the father of medicine and a polymath with his name given to an African journal of pure and applied mathematics.

The three pyramids of Giza and the Great Sphinx were all built in the fourth dynasty, within a century from the first pyramid.

The long history of Egyptian civilization is dominated by a single people with a common culture and language in sharp contrast to the Mesopotamian history. The big exception is the period between the Middle and New Kingdom (1650–1550 BC) when Egypt was ruled by a people from the west called Hyksos.

Time	Era/Language	Center	Note
3100 – 2690	**Early Dynastic Period** Ancient Egyptian	Memphis	Menes (Narmer) ~3200±100
2690 – 2180	**Old Kingdom** Ancient Egyptian	Memphis	Pyramids of Giza 2600–2500
2180 – 2050	**1st Intermediate Period** Ancient Egyptian	Herakleop. Thebes	
2050 – 1650	**Middle Kingdom** Ancient Egyptian	Thebes	Moscow papyrus 1800 Rhind papyrus 1650
1650 – 1550	**2nd Intermediate Period** Ancient Egyptian	Avaris Kerma	Dynasty of Hyksos Kingdom of Kush
1550 – 1070	**New Kingdom** Ancient/Late Egyptian, Nubian, Canaanite	Thebes Memphis	Hatshepshut –1460 Akhenaten –1330 Ramesses II –1210
1070 – 660	**3rd Intermediate Period** Late Egyptian	Tanis Bubastis etc.	Fragmented rule
660 – 330	**Late Period** Demotic Egyptian	Sais Mendes	Foreign rule Province of Persian Empire
330 – 30 BC	**Ptolemaic Egypt** Demotic Egyptian Greek, Berber	Alexandria	Alexander III –320 Ptolemy I Soter –280 Cleopatra –30 BC
30 – 640 AD	**Roman and Byzantine** Demotic Egyptian Coptic Egyptian	Alexandria	Augustus –14 AD Roman governors Hellenistic culture
640 – 970	**Arab Egypt** Arabic, Coptic	Fustat	Rashidun Caliphate –660
970 – 1170	**Fatimid Egypt** Arabic, Coptic	Cairo	Fatimid Caliphate
1170 – 1250	**Ayyubid Egypt** Arabic, Coptic	Cairo Aleppo	Saladin –1190
1250 – 1520	**Mamluk Egypt** Arabic, Coptic, Turkic, Circassian,	Cairo	Descendants of slave soldiers from Turkic and Caucasian regions

Timetable for Egypt from the fist dynasty 3100 BC to the Ottoman era 1520 AD.

The Egyptian mathematical tradition extends to 330 BC when Alexander the Great conquers the area. From that time on it becomes a part of the Greek tradition of mathematics. This period lasts till the Arabic era about 600 AD.

Egyptian writing dates back to c. 3200 BC with some 500 symbols that can represent both words and sounds. It existed in two versions hieroglyphs and hieratic script. Hieroglyphs were more formal and were used for inscriptions on stone walls whereas hieratic script had a more cursive style that was used by scribes on a daily basis. Decoding the script became possible with the Rosetta Stone, created in the Ptolemaic period 196 BC and rediscovered by a Napoleonic expedition to Egypt in 1799. It has the same text written with three different scripts. On top are hieroglyphs, in the middle is demotic script, successor to hieratic script and the bottom third is written with Greek letters.

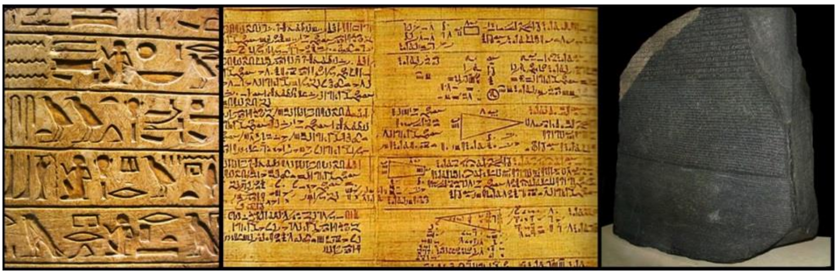
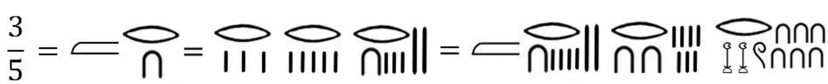


Fig 2.5.2 Hieroglyphs, Rhind papyrus in hieratic script and the Rosetta Stone.

The Egyptian numeral system for integers was like the Roman system but without subtractions. 𓁨 𓆐𓆐 𓂭𓂭𓂭 𓆼 𓍢𓍢 𓎆𓎆𓎆𓎆 𓏽𓏼 = 1.231.247, it was additive and based on ten.

When we see a fraction $3/5$ we think three fifths. The numerator numerates how many fifths there are and the denominator denominates what there are three of. You can’t add fractions with different denominators; it would be like adding apples and oranges. You must transform them into a common denominator like fruits before you add them. This is not how the Egyptians thought of fractions.

Fractions were written as sums of different unit fractions $(1/N)$. This can be done in many ways, $3/5$ equals both $1/2 + 1/10$ and $1/3 + 1/5 + 1/15$. Special procedures and tables were used to find a sum. Unit fractions were written with the r-sound hieroglyph shaped like a mouth except for some fractions that had their own symbol ($1/2$, $2/3$ and $3/4$).



Egyptian fractions may seem a bit quirky but they can be good for equal distribution. If a supervisor want to divide 3 units of food to 5 workers, he would first give each half a unit and then divide the remaining piece evenly, $3/5 = 1/2 + 1/10$. An alternative would be to use a binary expansion, $p/q = (N.b_1b_2b_3\dots)_2$, give each worker N pizzas, halve the remaining pizzas and give each worker b_1 halves, halve the remaining pieces and so on.

In the Old Kingdom they actually had this system with hieroglyphs from the Eye og Horus. Horus was the sky god, depicted as a falcon with the right eye linked to the sun god Ra and the left eye linked to the god Thot. Legend says that Set and Horus were fighting for the throne after Osiris' death. Set tore the left eye to pieces but Thot restored the eye and Horus offered the eye to his father Osiris in hope of restoring his life. Binary fractions were replaced by Egyptian fractions in the Middle Kingdom.

$3/5$ can not be written as a finite sum of binary fractions $1/2^n$. A method to get such sums was first described in Fibonacci's book "Liber Abaci" from 1202. The method is called a greedy algorithm, it begins with the biggest possible unit fraction and continues greedily with what remains. In modern mathematical notation:

Greedy algorithm for Egyptian fractions

$$\frac{x}{y} = \frac{1}{\lceil y/x \rceil} + \frac{(-y) \bmod x}{y\lceil y/x \rceil}$$

The last term is repeatedly and similarly expanded.
ceiling function, floor function and modulo operator:
$\lceil x \rceil$ = smallest integer not less than x.
$\lfloor x \rfloor$ = largest integer not greater than x.
$p \bmod q \in \{0,1,\dots,q-1\} \cap \{p + nq | n \in \mathbb{Z}\}$.

$$\frac{8}{11} = \frac{1}{2} + \frac{5}{22} = \frac{1}{2} + \frac{1}{5} + \frac{3}{110} = \frac{1}{2} + \frac{1}{5} + \frac{1}{37} + \frac{1}{4070}$$

Egyptians used other methods. The most famous mathematical document from ancient Egypt is the Rhind Mathematical Papyrus (RMP), discovered by Alexander Rhind from Scotland. It is written in hieratic script around 1650 BC. The first part contains reference tables for Egyptian fractions. One table shows $2/n$ for odd n up to 301 as sums of unit fractions.

Harmonic hangover with a constant appearance

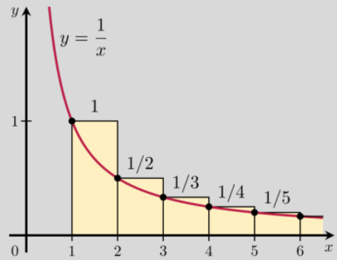
There is a series of numbers called harmonic numbers H_n.
In semi-Egyptian notation they are: $\bar{1}$, $\bar{1}\ \bar{2}$, $\bar{1}\ \bar{2}\ \bar{3}$ …

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

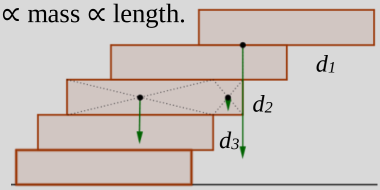
$$\int_1^{n+1} \frac{1}{x} dx \quad < H_n < \quad 1 + \int_1^{n} \frac{1}{x} dx$$

$$\ln(n+1) \quad < H_n < \quad 1 + \ln(n)$$

$$\lim_{n\to\infty} H_n = \infty$$



The numbers occur in a stack of blocks with maximal hangovers d_n. Blocks of length 2 have $d_1 = 1$. Torque $(\bar{F} \times \bar{r})$ from left part of a block equals that from the right part and from the center of mass from blocks above. Force of gravity ∝ mass ∝ length.



$$(2 - d_n) \cdot \frac{2 - d_n}{2} = d_n \cdot \frac{d_n}{2} + (n-1) \cdot 2 \cdot d_n \Rightarrow d_n = \frac{1}{n}$$

The total hangover for n blocks $H_n \sim \ln n \to \infty$ but $(H_n - \ln n)$ is strictly decreasing with a limit γ that is called the Euler-Macheroni constant.

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) = 0.577215664901532 \ldots$$

The constant occurs frequently in number theory and analysis, probably non-algebraic but still not even proved to be irrational. If it is rational then the minimal denominator has been shown to be larger than 10^{242080}.

Constant	Introduced	Proof irrational	Proof non-algebraic
π = 3.14159…	Egypt/Babylon	1761 J. Lamberth	1882 C. Lindemann
e = 2.71828…	1618 J. Napier	1737 L. Euler	1873 C. Hermite
γ = 0.57721…	1734 L. Euler	Not yet	Not yet

There are still unsolved puzzles surrounding Egyptian fractions. The famous mathematician Paul Erdös (1913-1996) made the following conjecture with E.G.Straus: every fraction $4/n$ is a sum of three unit fractions.

The Rhind papyrus also contains 84 problems. Some of them corresponds to linear equations, like the 32nd: $x + 1/3 \cdot x + 1/4 \cdot x = 2$. Egyptians from 1650 BC could solve some of our introductory equations but the concept of periodic fractions would not exist in their number system.

Notations for equations and variables were a much later inventions but the papyrus contains signs for addition 𓂻, subtraction 𓂽 and square root ⌈. Multiplication was made by implicit use of binary representation and the distributive law: $(a + d) \cdot c = ac + dc$. The same method worked for division to get the quotient (q) and remainder (r), $a/b = q + r/b$. It could also give the result as an Egyptian fraction or at least an approximation.

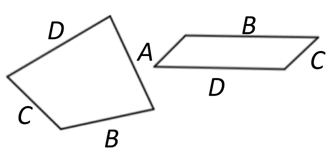
11×35		
1	35	✓
2	70	✓
4	140	
8	280	✓
$(1011)_2 = 385$		

5025/365 (Egyptian)		
1	365	✓
2	730	
4	1460	✓
8	2920	✓
$q = 13$ $r = 280$		
2/3	243 1/3	✓
1/10	36 1/2	✓
1/2190	1/6	✓
13+2/3+1/10+1/2190		

5025/365 (Binary)		
1	365	✓
2	730	
4	1460	✓
8	2920	✓
$q = 13$ $r = 280$		
1/2	182.5	✓
1/4	91.25	✓
1/8	45.625	
$(1101.110\ldots)_2$		

The division corresponds to problem 66 in RMP where a quantity should be distributed evenly over 365 days.

Egyptian geometry illustrates how they thought about mathematics. The driving force was to solve practical or religious problems. Formal proofs were not used and a description of how to calculate a geometrical quantity might not be correct just good enough to serve its purpose. The area of a quadrilateral field was $\frac{1}{2}(A + C) \cdot \frac{1}{2}(B + D)$. If the answer was too wrong they would probably just use another method.



Problem 48 in the Rhind papyrus approximates a circle with an octagon to get $\pi \approx 256/81 \approx 3.16$.[7]

Angles were measured in a unit called seked. It was based on the royal cubit, approximately the length of a forearm which was divided into seven palms. Each palm was divided into four digits. Slopes were given as number of horizontal palms per vertical cubit. The slope of the Great Pyramid in Giza is 5½ seked $= \arctan(7/5.5) \approx 51.8°$.

A driving force behind mathematics in early civilizations was astronomy, to track the motions of the sun, the moon and the planets. These were often linked to divine figures. It was important to create calendars to record events and keep track of the seasons for agriculture and religious ceremonies. The cardinal directions were used when pyramids were built.

The Egyptians were the first to adopt a solar calendar, maybe as early as in the Old Kingdom. In the Middle Kingdom they had a calendar with 365 days based on the observation of Sirius rising above the horizon at sunrise. Since a proper year is 365.25 days stellar events and seasons would wander through the calendar in a cycle of 1460 years. They were aware of this and used leap months to compensate. A regular year was divided into 12 months of 30 days and an extra 5 days at the end to celebrate the gods. Ptolemaic rulers decided in 238 BC that every 4th year should be 366 days long. This reformed calendar is still used in parts of Egyptian society.

In a contest between Egyptian and Babylonian mathematics; which one of them would be the winner? There are many similarities. Both used verbal language to describe mathematical problems and procedures. Both were focused on practical applications without formal proofs of statements. Mathematics as we define it was an invention of the Greeks. But the winner, without doubt, would be Babylonian mathematics. Their astronomy became much more advanced and their number system with its positional system was more suited for calculations. The roots to the Greek development in astronomy and mathematics are to a large extent Babylonian.

2.6 Indian Mathematics

Fig. 2.6.1 Aum symbol, modern Hindu temple, Taj-Mahal and dharma symbol.

The first civilization on the Indian subcontinent is located at the Indus river system in the border region between India and Pakistan. Excavation of Harappa in the north and Mohenjo-Daro in the south started in the 1920s and is still ongoing.

The Indus Valley Civilization has a long prehistory but its central period with the largest extent was 2600–1900 BC and its peak population could have been as large as five million. The language or languages they used are long extinct and their relation to other language families is uncertain. Poor knowledge of the language, only short inscriptions found and no bilingual texts makes the written language hard to decipher. Their weights and measures were in multiples of 10 starting from 1, 2 and 5.

The Indus Valley Civilization was later overtaken by Indo-Aryan people from the north. Their language was the origin of Sanskrit that belongs to the Indo-European language family that comprises almost all European languages and also the Indo-Iranian languages spoken from Iran to northern India. The area in between these language areas is dominated by Turkic languages.

Oral traditions from as far back as 2000 BC were written down in Sanskrit during the Vedic period (1500–500 BC). Vedas is a large collection of texts from this period, veda is a Sanskrit word meaning knowledge. A history somewhat similar to the Jewish Tanakh and the Christian Bible with their many texts from different ages, presented at times in outdated languages, Hebrew and Latin, preserved for religious texts and liturgy.

The Vedic Sanskrit language was grammatically analyzed and standardized by Panini some time 600–400 BC. His methods were very advanced and formalized almost mathematical in nature. Paninis grammatical analysis resembles modern work in linguistics and computer science. In the book "The Crest of the Peacock" by Georg Joseph it is argued that the algebraic nature of Indian mathematics is a consequence of the structure of the Sanskrit language. Panini is a true giant in the history of cultural development.

The earliest texts with mathematical content in the Hindu civilization are the Shulba Sutras from the Vedic period. They contain geometrical descriptions for construction of sacrificial altars and are part of a larger set of text about religious ceremonies and rituals which in turn are part the Vedas. The Vedas have a role in Hinduism comparable to the Bible and the Quran, believed by their worshippers to be of divine origin.

Baudhayana is the author of one of the earliest texts in the Shulba Sutras. Dating is uncertain and could be anywhere from 800 to 600 BC. It contains rules for sacrifices, rituals and geometrical statements, like the following: दीर्घचतुरश्रस्याक्ष्णया रज्जु: पार्श्र्वमानी तिर्यग् मानी च यत् पृथग् भूते कुरूतस्तदुभयं करोति "A rope stretched along the length of the diagonal produces an area which the vertical and horizontal sides make together".

Pythagoras' theorem was apparently known in both Babylonian, Egyptian and Indian mathematics several centuries before Pythagoras. Baudhayanas work was later expanded by Apastamba in another Shulbasutra with a proper proof of Pythagoras' theorem. Baudhayana is the second person in the very long chronological listing of mathematicians in the MacTutor archive. Preceeded only by Ahmes from the Rhind Papyrus. Ahmes own comment in the papyrus says that he is not the author but merely a scribe collecting material from an earlier work of about 2000 BC. It is quite reasonable to say that Baudhayana is the first mathematician we know of by name.

The Shulbasutras of Baudhayana contains geometric solutions to linear equations and some quadratic equations, how to make a circular altar of the same area as a square and finding the length of the diagonal in a square. He had several bad approximations to π and a good approximation of $\sqrt{2}$, much better than would be need for the construction it was intended for. The Sanskrit expression for this is *dvi-karani*, "that which produces two"

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \approx 1.4142157$$

This approximation is as good as but much later than the one mentioned from Mesopotamia. A possible link between Indian and Mesopotamian knowledge is the Vedanga Jyotisa, the earliest known text from India dealing with timekeeping, the movements of the sun, moon and planets. Its author is Lagadha but the dating is very uncertain. It describes a winter solstice that matches the period 1400 BC but is believed to be of a later date, possibly based on traditions going back to 700 BC. The text shows clear parallels with similar texts from Mesopotamia.[8]

Astrology which still has a big influence in Indian society is of a later date, based on Greek astronomy derived from Babylonian astronomy and cultural influence starting with Alexander the Great 327 BC.

After the Vedic period of Hinduism came Jainism with a tradition of non-violence (ahimsa) and self-discipline to reach liberation and Buddhism with the teachings of Buddha. They all had elaborate cosmologies and myths with enormous numbers, infinities and time spans up to 2^{588} years. These ideas were very sophisticated and speculative but based on metaphysics and mythology rather than mathematics. Similar ideas would appear in the 19th century with Cantors theory of cardinal and ordinal numbers and 20th century computer science with computable and non-computable numbers.

The Yajurveda (1200–900 BC), a text with mantras or sacred formulas to use in rituals of sacrificial prayers has names for each positive power of 10 up to 10^{12} (1 Paraardha) and for numerical infinity (Purna). The Yajurveda states "subtract purna from purna and you get purna", like removing even integers from all integers and you still have infinity. Ramayana, a Sanscrit epic poem (500–200 BC) about the duties in relationships and how to be a good person has a number system with names for numbers up to 1 mahaugha=10^{60}.[8a] Rama's bridge, the land bridge between India and Sri Lanka was built by 10^{69} monkey soldiers.[8b] It was above sea level until AD 1480 when it broke in a cyclone. Other large numbers are Dhvajagranishamani 10^{421} and Asam-khyeya 10^{x}, with x as big as $7 \cdot 2^{103}$, depending on translation. Asamkhyeya meaning 'innumerable' or 'incalculable' is also the title of Vishnu and Shiva. The Avatamsaka sutra of Mahayana Buddhism which describes a cosmos of infinite realms upon realms containing one another has a number 10^{x} with $x = 7 \cdot 2^{122}$. Jaina mathematicians classified numbers as enumerable, innumerable or infinite and they had five different types of infinities.

Big numbers in the west where tiny in comparison. The ancient Greeks had the number myriad (10 000) and their largest number was a myriad myriad (10^{8}). In "The Sand Reckoner" Archimedes tried to find an upper bound for the number of sand grains to fill the universe. His biggest number was $(\text{myriad}^{\text{myriad}})^{\text{myriad}}=10^{x}$ with $x = 8 \cdot 10^{16}$.

A modern version of the big number name game are googol and googolplex, invented in 1920 by 9-year old Milton Sirotta, nephew of American mathematician Edward Kasner (1 googol=10^{100}, 1 googolplex=10^{googol}). The company name Google originated as a misspelling of googol. The Google corporate headquarters is called the Googleplex.

Big numbers part 1

To generate large numbers from one could use multiplication $a \cdot b$ or even better exponentiation a^b which is neither commutative $a^b \neq b^a$ nor associative $(a^b)^c \neq a^{(b^c)}$. a^{b^c} means $a^{(b^c)}$ (a right-associative operator). A left-associative convention would collapse the tower $(a^b)^c = a^{bc}$. The biggest number with three digits and common arithmetic is 9^{9^9}. The top level is most important for growth.

$$100^{2^2} = 10^8 \qquad 2^{100^2} = 10^{\lg 2 \cdot 10^4} \approx 10^{3 \cdot 10^3} \qquad 2^{2^{100}} = 10^{\lg 2 \cdot 10^{100 \lg 2}} \approx 10^{4 \cdot 10^{29}}$$

A notation for large numbers that avoids high towers is Knuth's up-arrow notation, named after Donald Knuth, computer scientist, inventor of algorithms, designer of TEX typesetting and much more.

$$\underbrace{A + \cdots + A}_{B} = A \cdot B \qquad \underbrace{A \cdot \ldots \cdot A}_{B} = A^B = A \uparrow B \qquad \underbrace{A \uparrow \cdots \uparrow A}_{B} = A \uparrow\uparrow B = A \uparrow^2 B$$

$$\underbrace{A \uparrow^n (A \uparrow^n (\ldots \uparrow^n A) \ldots)}_{B} = A \uparrow^{n+1} B \qquad 3 \uparrow^3 3 = 3^{\cdot^{\cdot^{3}}} \}3^{27} \text{ levels}$$

The operation $\uparrow\uparrow$ is called tetration. To get a sense of the size of these numbers try calculating $4 \uparrow^4 4$ or look at "Ridiculously huge numbers" by D. Metzler on YouTube. A close relative is hyperoperation $\mathrm{H}_n(a, b)$.

$$\mathrm{H}_0(a, b) = b + 1 \quad \mathrm{H}_1(a, b) = a + b \quad \mathrm{H}_2(a, b) = a \cdot b \quad \mathrm{H}_n(a, b) = a \uparrow^{n-2} b$$

Simple problems can generate surprisingly big numbers. Let's look at our last exercise from chapter one:

$A + B + C + D = 1$
$B = 1\%$ of A
$C = 1$ppm of B
$D = 1‰$ of C

Pick the periodic digits with a function:
$\mathrm{D}: \mathbb{Q}^+ \to \mathbb{Z}$
$\frac{p}{q} = a_1 \ldots a_i . b_1 \ldots b_j \overline{c_1 \ldots c_k} \mapsto c_1 \ldots c_k$

$$\mathrm{D}(\mathrm{A}) \approx 10^{25014015913} \approx 10^{10^{10.4}} \approx 10 \uparrow\uparrow 3$$

The Googolplex $10^{10^{100}}$ is bigger but not big $10 \uparrow\uparrow 4 = 10^{10^{10,000,000,000}}$ which is much much smaller than $10 \uparrow\uparrow\uparrow 2 = 10 \uparrow\uparrow 10$.

In the next part on big numbers we will look at notation and problems that leads to much bigger numbers.

The greatest legacy of early Indian mathematics is not the big numbers but the small numbers 1,2,3,4,5,6,7,8,9 and 0. The Hindu-Arabic number system is superior for calculations compared to the historic alternatives. Our digits can be traced to Brahmi numerals from 300 BC. A special symbol for zero and a positional system came later. Jain mathematicians used the Sanskrit word śūnya meaning void for zero. After a long series of translations*, śūnya* Sanskrit→Arabic→Venetian→Italian→French→English became *zero*. The Indian numerals can be seen in the Bakshali manuscript, seventy pieces of birch bark found near the village of Bakshali in the Peshawar district in present day Pakistan. Radiocarbon dating reveal a wide spread in ages for different parts, the oldest pieces are from 300–400 AD. The manuscript was found in 1881 and is presently at the Bodleian Library in Oxford. In the manuscript a dot is used both for the number zero and to represent an unknown number.

Fig 2.6b Bakshali manuscript and numerals from the manuscript.

The Indian numbers spread westwards, first to Persian mathematicians in India, and from them to Arab mathematicians and from them to Europe via Arab merchants. The Syrian bishop Severus Sebokht wrote in 662 AD:

"I will omit all discussion of the science of the Indians, ... , of their subtle discoveries in astronomy, discoveries that are more ingenious than those of the Greeks and the Babylonians, and of their valuable methods of calculation which surpass description. I wish only to say that this computation is done by means of nine signs. If those who believe, because they speak Greek, that they have arrived at the limits of science, would read the Indian texts, they would be convinced, even if a little late in the day, that there are others who know something of value."

The origin of the Bakshali manuscript is of a much later date than the era of Alexander III of Macedon with Hellenistic influence but the content is still very “Indian” in the same sense that we do not call Greek mathematics “Babylonian”.

The Bakshali manuscript is vital for understanding Indian mathematics from 200 BC to 500 AD, the time of Aryabhata I.

The manuscript contains calculation with fractions using a notation without bar but numerator above denominator and the sign + for subtraction. Mixed fractions like 3½ were represented with 3, 1 and 2 arranged vertically. Use of a horizontal bar in fractions is mentioned by a Muslim mathematician Al-Hassār from Morocco in the 12^{th} century. The fraction bar was introduced to Europe by Fibonacci in his book "Liber Abaci" from 1202.

The manuscript contains math problems that correspond to systems of linear equations and a method for finding an approximation to the square root:

To calculate $\sqrt{x}$: Let N^2 be the nearest square to x $y = x - N^2$ $\sqrt{x} \approx N + \frac{y}{2N} - \frac{y^2}{8N^3 + 4Ny}$	$x = 95 \rightarrow N = 10, y = -5$ $\sqrt{95} \approx 10 - \frac{1}{4} - \frac{1}{328} = 9.7469\ldots$ $\sqrt{95} = 9.7467\ldots$

Interesting mathematical work prior to the Bakshali manuscript was made by Pingala (~300–200 BC). He was a scholar in music and Sanskrit that came up with Fibonacci numbers, the Pascal triangle without knowing of the binomial theorem (explained later) and a binary numeral system. An important contribution of Jain Mathematics 400BC–200CE was to free mathematics from its religious constraints. We will return later to Indian mathematics of the "Classical Period" (400–1600).

2.7 Chinese Mathematics

Fig 2.7.1 Chinese architecture, landscape, Ming porcelain and abacus

The abacus has been a practical tool for calculation in China for many centuries and up to the present day but it was used in Mesopotamia, Egypt and Greece long before China.[9] Among the ancient river valley civilizations; the Nile, Euphrates and Tigris, Indus and the Yellow River (Huang-He), the Chinese was the one that had the biggest geographical barriers to the others.

Geography and at times self-imposed withdrawal resulted in independent mathematical development, like the reinvention of the abacus. The abacus illustrates the use of a positional decimal system. The origin of which could very well have been chinese with Indian mathematics adding the zero.

As in other early civilizations the driving force behind mathematics was agricultural calendars, astronomy, trade, surveying and other practical tasks. Knowledge of Chinese mathematics before 250 BC is fragmentary, the archaeelogical findings are not as as rich as from Egypt and Mesopotamia. With the Qin dynasty comes expansion south and imperial rule where mathematics is put into the service of the empire. There were periods of advance, stagnation and decline but mostly continuity with emphasis on procedures for solving certain problems and reproduction of knowledge. The imperial examination system for the administration with roots from the Han dynasty and fully established during the Tang dynasty became a central part of a common intellectual, cultural and political life. It remained in place until 1905. This section will cover Chinese mathematics up to about 500 AD.

Time	Ruling era	Notes
2070 – 1600 BC	**Xia dynasty**	Yu the Great, legendary founder
1600 – 1050 BC	**Shang dynasty**	Center Yinxu, writing on oracle bones
1050 – 256 BC	**Zhou dynasty**	Western and Eastern Zhou Spring and Autumn period 770–480 BC Oldest part of the Great Wall Warring states 475–221 BC
221 – 206 BC	**Qin dynasty**	Qin Shi Huang, etymology Qin→China city Xianyang, burning of books 210 BC
206 – 220 AD	**Han dynasty**	1st golden age, name to Han people Adoption of Confucianism
220 – 280	**Three Kingdoms**	Wei, Shu and Wu, period of warfare
265 – 420	**Jin dynasty**	Western Jin, war of 8 princes, Eastern Jin
304 – 439	**Sixteen Kingdoms**	Northern China divides ethnically
420 – 589	**South/North Dynasties**	Six Dynasties period (220-589)
581 – 618	**Sui dynasty**	Unification after north/south division
618 – 907	**Tang dynasty**	Capital Chang´an (Xian), 2nd golden age
907 – 1125	**Liao dynasty**	Conquest dynasty from north of China
907 – 960	**Five dynasties**	Era of political upheaval
960 – 1279	**Song dynasty**	Unification, strong government, growth
1271 – 1368	**Yuan dynasty**	Mongols, established by Kublai Khan
1368 - 1644	**Ming dynasty**	Capital Beijing, Great Wall expanded
1644 – 1911	**Qing dynasty**	Manchuria dynasty, current extension
1912 – 1949	**Republic of China**	Xinhai revolution, civil war
1949 –	**People's rep. of China**	Mao (–1976) , Communist Party

Early examples of mathematics from China can be seen on Oracle bone script from the Shang dynasty from 1600 to 1050 BC. The text which was incised on animal bones and turtle shells is the oldest ancestor to modern Chinese, Japanese and Korean scripts. The numeral system was based on ten and additive so there was no need for a zero.

With the invention of the counting board in the 4th century BC came a second form of numerals. The counting board consisted of a checker board with rows and columns. Numbers were represented by rods made from bamboo or ivory. The rightmost column contained units, the next contained tens and so on which made it a true positional system but without zero. Two types of digits were used to avoid misreading numbers with several digits. Rods of different color were used for positive and negative numbers. Counting boards spread to Japan, Korea and Vietnam and were used up to the 16th century

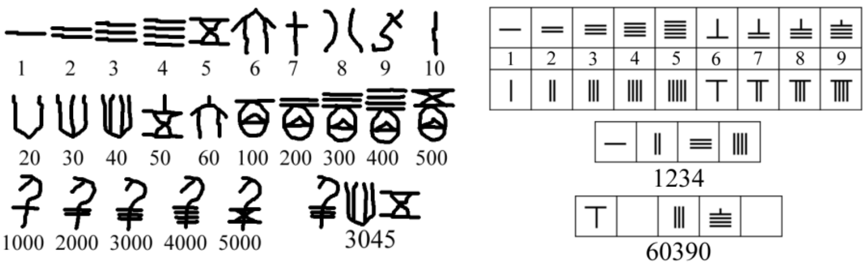
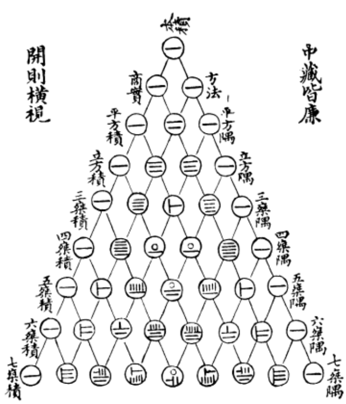


Fig. 2.7.2 Chinese numerals on oracle bones and counting board.

The first mathematical developments in China had no predecessors, at a later stage there may have been some influence between Chinese and Indian cultures. The Chinese discoveries were mostly of their own making; some for the first time later to be rediscovered by the rest of the world, others already discovered elsewhere to be rediscovered in China. Some examples: negative numbers, place value decimal system, binary system, Pascal’s triangle and the binomial coefficients, proof of Pythagoras’ theorem, approximations of π, trigonometry, solution to systems of linear equations, modular arithmetic.



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Fig. 2.7.3 Pascal's triangle in two versions.

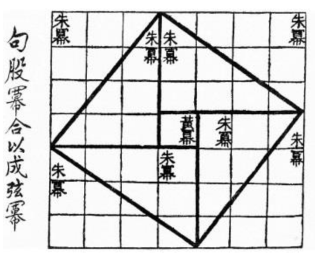
Much of early Chinese mathematics is presented in texts that were and still are central in Chinese cultural heritage.

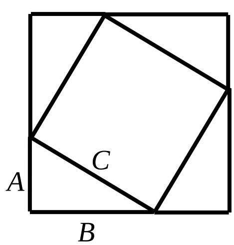
Titles are followed by their pinyin version, the system used for turning Chinese pronunciation into western script. Sometimes there are alternative forms due to a previous system for phonetic transcription (Wade-Giles) and sometimes you see tonal marks on pinyin vowels to avoid ambiguity. Mandarin has five different tones: mā, má, mǎ, mà, må.

Book of changes (Yi Jing) is the oldest of the Chinese classics, with origins from ~1000 BC followed by millennia of commentary and interpretation. It has some mathematical content but its importance lies in its influence on philosophy, religion, literature and life in general. It started as a book on rituals and transformed to a major work of philosophy with the ***Ten Wings*** commentaries traditionally ascribed to Confucius (551–479 BC).

Mathematics had a high standing in Chinese culture. It was one of the Six Arts (liù yì) with roots in Confucian philosophy that formed the basis of education during the Zhou dynasty, others were music and calligraphy. To be a gentleman was to master the Six Arts.

Zhou Shadow Gauge Manual (Zhou Bi Suan Jing) is the oldest complete surviving mathematical text, compiled between 100 BC and 100 AD. Its main focus in on astronomy and how to measure positions on the celestial sphere but it also gives a clear view on mathematical thinking of the time and the importance of being able to deduce and generalize. These methods are at the center of modern mathematical development. The text also contains the Gougu rule, the Pythagorean Theorem and a visual geometrical proof.





Area of large square:

$$(A+B)^2 = 4\frac{AB}{2} + C^2$$

$$\Downarrow$$

$$A^2 + B^2 = C^2$$

The Nine Chapters on the Mathematical Art (Jiu Zhang Suan Shu) is the most famous mathematical text. It is a practical handbook with 246 problems and procedures for their solution. The book's importance for Chinese mathematics is comparable to the role of Euclid's *Elements* in Europe. It was composed by generations of scholars during the Zhou dynasty.

Liu Hui edited and expanded the book in 263 AD. In the preface he writes:[10]

"When Zhou Gong set up the rules for ceremonies, nine branches of mathematics were emerged, which eventually developed to the Nine Chapters of the Mathematical Art. Brutal Emperor Qin Shi Huang burnt books, damaging many classical books, including the Nine Chapters. Later, in Han Dynasty, Zhang Cang and Geng Shou Chang were famous for their mathematical skills. Zhang Cang and others re-arranged and edited the Nine Chapters of Mathematical Art based on the damaged original texts."

Some of the content in the nine chapters are as follows:

Ch.1 Land surveying with area problems, arithmetic with fractions and Euclidean algorithm for the greatest common devisor.

Ch.2 Trade with proportions and percentages.

Ch.4 Extractions of square and cube roots with notions of limits and infinitesimals.

Ch.7 A complicated method to solve linear equations $ax + b = c$.

Ch.8 Problems with systems of linear equations and a method similar to Gaussian elimination.

Ch.9 Right-angled triangles, similar triangles and Pythagoras' Theorem.

In a comment by Liu Hui where he tries to find a formula for the volume of a sphere, he writes *"Let us leave the problem to whoever can tell the truth"*. The problem was already solved by Archimedes (287–212 BC). *Nine Chapters* served as a textbook in China and parts of eastern Asia for more than a millennium, some of its formulas appear in 16th century Europe.

Among early Chinese mathematicians, **Liu Hui** (220–280) is one of the greatest. He emphasized generalizations and proof, incorporated negative numbers in arithmetic and used the limit concept. He used recurrence and interpolation to approximate π based on regular polygons with $3 \cdot 2^n$ sides. His approximation for pi 3.1416 was the best of its time. Zu Chongzhi used the method on a 12288-gon and held the record for 900 years.

Xiahou Yangs Mathematical Journal from c. 450 AD contains arithmetic with positive and negative powers of ten but still no specific symbol for zero.

2.8 Mayan Mathematics

Fig. 2.8.1 Maya long date, geography, Aztec calendar, Chichenitza, Maya numerals

The mathematics of old American civilizations like the Mayan civilization, 250AD–650 (green area) and the Aztec empire 1427AD–1519 (pink area) developed completely independent from the old world. For all practical purposes they can be seen as pre-Greek even if they were more recent. The red object is a colorized version of the Aztec calendar stone that incorporates Mayan agricultural and ritual calendars.

The Mayan numerals were the most sophisticated number system of pre-Columbian America. The Mayans had a positional system based on twenty with just three different symbols. A dot for one, a bar for five and a special symbol usually a shell for zero. The shell symbolized no content. Twenty was a very important number to the Mayans. The words for “twenty” and “human being” have the same root in Mayan language. Numbers were written from top to bottom with higher powers at the top. They also had head glyphs for numbers which shows the importance of practical notation for mathematical calculation.

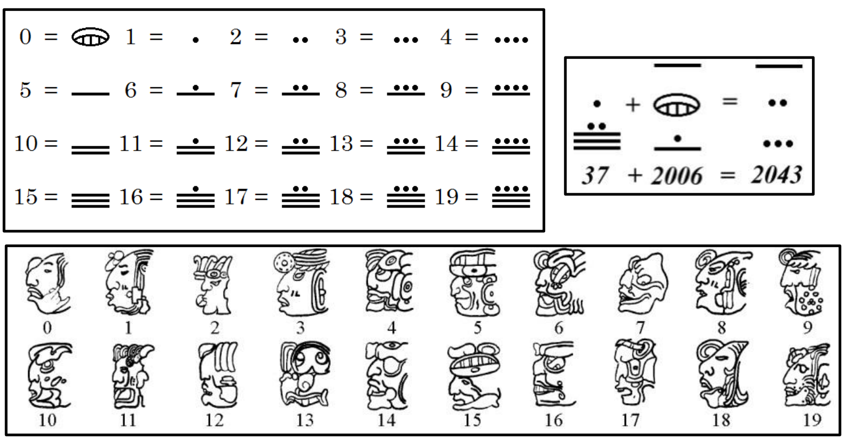


Fig 2.8.2 Mayan numerals

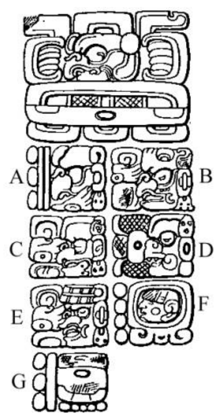
The Mesoamerican civilizations from early Zapotec and Olmec cultures to later Aztec had two cyclic calendars, a ritual calendar of 260 days and a solar calendar of 365 days. The sacred calendar (Tzolkin) had 20 weekdays and 13 weeks, the other calendar (Haab) had 18 months of 20 days and five nameless dangerous days at the end when the border to the underworld stood open.[11] Together they formed a cycle called the Calendar Round with 18980 days, the least common multiple (LCM) of 260 and 365. This period of 52 years is still used in Guatemala. To record history over long periods they used the Long Count calendar. It is a linear system that counts the number of days since the creation of the current world in Mayan mythology. Long Count dates can be seen on monumental stone slabs, a shorter form was still used when conquistadors arrived in 1517. Translated to the Gregorian calendar the year zero in the Long Count was on August 11, 3114 BC.

Creation mythologies tend to start with orally transmitted stories long before they are written down, like the Hebrew Bible. Rabbinic estimates in the 12th century put the creation date to 3761 BC, Anno Mundi (AM). The Hebrew calendar used in religious ceremonies counts the years from this date. The oldest long date inscriptions found is from 36 BC, millennia after year zero. Somebody may have done what the rabbis and the Irish bishop James Ussher did. He claimed in 1650 that the world was created on October 22, 4004 BC. His claim was based on Biblical accounts of the age of direct descendants of Adam and Eve. For some reason the early descendants were all very old, in the range 700–900 years.

Quirigua stela from Quirigua with
date 13.0.0.0.0 and 4Ahau 8Kumku

A.B.C.D.E is Long Count date
E: Kin (= 1 day)
D: Uinal (= 20 kin)
C: Tun (= 18 uinal= 360 days)
B: Katun (= 20 tun)
A: Baktun (= 20 katun $\approx$ 394.259 years)
$18(20^3 A + 20^2 B + 20C) + 20D + E$
days since creation in 3411 BC.

F.G is from the Calendar Round
first date from 260-day calendar (Tzolkin)
then date from 365-day from calendar (Haab).[12]



Mayan mythology can be read in a book called Popul Vuh “Book of the People”. The text is based on oral recitations written down by early conquistadores. The gods created three failed worlds before they succeeded and placed humans in it. The last world ended after 13 baktuns or 5125.36 years. The date 13.0.0.0.0 in the present era occurred on December 21, 2012.[13]

Mayans named time spans as large as $18 \cdot 20^7$ days ≈ 63 million years. The concept of a linear calendar counting days from a specific date is also used today by astronomers. Julian Day Numbering (JDN) starts from January 1, 4713 BC in the proleptic Julian calendar. The Julian Date as I write this is 2,457,337.13.

Maya civilization made excellent astronomical observations to support their calendars and for their astrology, a solar year 365.242 days (true value 365.242198) and a lunar month 29.5302 days (true value 29.53059).[14]

The idea of place value was invented at least four times and the concept of zero was discovered at least two times. Mesoamerican civilizations and Maya were involved in both. Cultural disconnectedness of ancient times made mathematical development a matter of geography.

2.9 Greek Mathematics

Fig 2.9.1 Ancient Greece, Parthenon, ‘The School of Athens’ fresco in the Vatican

The cultural roots of Ancient Greece goes back to the Minoan civilization on Crete (3650–1450 BC) and the Mycenaean civilization (1600–1100 BC). The term Minoan is a late invention based on the mythical king Minos of Knossos who had a labyrinth constructed by Daedalus and his son Icarus. Every nine years he sent seven boys and seven girls into the labyrinth to be eaten by a Minotaur at its center.

The Minoan language is a great mystery. Its relation to other languages is unknown. The oldest script of the language is called Cretan hieroglyphs, it was later complemented by Linear A. Both scripts are still undeciphered.

The Mycenaean language was an early form of Greek, part of the Indo-European group with a script called Linear B. Linear B has been deciphered and appears to have been a script used solely for administrative purposes by a limited group. Literacy in wider circles became more natural with the adoption of an alphabet c. 800 BC. The Greek alphabet is derived from the Phoenician alphabet, the oldest known alphabet. It was used 1200–150 BC by the Phoenicians from the eastern coast of the Mediterranean. They were a trading people with colonies around the Mediterranean, they spoke a Semitic language. Their alphabet was derived from Egyptian hieroglyphics with 22 consonants written from right to left. Phoenician merchants spread their alphabet along the Phoenician trading routes. One of its descendants is the Aramaic alphabet that has spawned both modern Arabic and Hebrew letters.

The Mycenaean civilization is named after the historic city of Mycenae on Peloponnesus. The kingdoms of Mycenae were destroyed in a long series of internal wars; disruption was so great that literacy was lost in the period called the "Greek Dark Ages" (1100–800 BC). Greek culture survived but poverty and chaos lead to emigration. New communities were established on the shores of Asia.

Dim and mythical memories of the Mycenaean period can be seen in the epic poems the *Iliad* and the *Odyssey* attributed to Homer. Attributed, since it is not known if Homer really existed as a historic person. Our knowledge of him is illustrated by the classic joke: "The poems of Homer were not written by Homer, but by another man of the same name". The *Iliad* and the *Odyssey* probably originates from oral traditions, describing a heroic past when gods still walked among men and played an active role. The poems are given in hexameter, a rhythmic prose that facilitates oral storytelling. The poems were put into writing and achieved a final fixed form some time before 700 BC.

The *Iliad* tells the story of a Mycenaean military expedition to the city of Troy in Asia Minor. The main characters are Agamemnon, king of Mycenae and Achilles, the tragic hero. Important parts of the plot are a Trojan horse and Achilles' heel. The *Odyssey* is a sequel to the *Iliad* that describes the journey home from the Trojan war of Odysseus and his men. Their journey on the Aegean Sea takes them ten years and many adventures in strange lands before Odysseus finally reach his wife Penelope on Ithaka. When Odysseus returns after twenty years he is assumed dead and finds Ithaka besieged by unruly suitors eager to take control of Ithaka and Penelope.

Odysseus' long and erratic journey back home from the war of Troy inspired the title of this book, the journey on the Aegean Sea like a wandering in a mathematical landscape. Had it been written in Swedish, my native language, I would have used the word 'irrfärd' which translates to odyssey. The journey may be erratic with many adventures on the way but like the odyssey of Odysseus there is a goal; to find a deeper understanding of the problems posed in the introductory chapter.

The epic poems are part mythical, part historical. Excavations in 1868, south of the Dardanelles in Turkey by Heinrich Schliemann revealed a series of cities built in succession from 3000 BC to 500 AD. One of the archaeological layers called Troy VII from 1300–1200 BC is believed to be the city described in Homer's Iliad.

Greek mythology, the origin and nature of the world, the gods and deities and religious rituals is presented in the poems *Theogony* and *Works and days*, written by Hesiod 700 BC. The poems show influence from Mesopotamian myths adopted and spread by the Hittite empire of Anatolia (1600–1178 BC). The poems contain stories about Zeus the ruler of Gods at Mount Olympus, Prometheus the creator of mankind and Pandora the first woman, created by Athena the goddess of wisdom, mathematics etc.

Time	Period	Note
3650 – 1450	Minoan	On the island of Crete, palace at Knossos Volcanic eruption of Thera, 1600–1500 BC
1600 – 1100	Mycenaean	First civilization on mainland Greece
1100 – 800	Dark Ages	Colonization of Asia minor
800 – 510	Archaic	First Olympic games 776 BC City-states, rise of Sparta and Athens Colonies along Mediterranean and Black Sea
510 – 323	Classical	Greco-Persian wars 499–449 BC Athenian Empire (Delian League 477 BC) Pericles (495–429), builder of Parthenon Peloponnesian war (431–404), Athens/Sparta Philip II of Macedonia rules Greece, 338 BC Alexander the Great builds an empire
323 – 146 BC	Hellenistic	Death of Alexander 323 BC at age 32 Ptolemaic dynasty in Egypt (305–30 BC) Seleucid dynasty in east (312–63 BC) with centers in Alexandria and Antioch
146 – 330 AD	Roman	Roman victory in battle of Corinth 146 BC
330 – 1453	Byzantine	Constantine makes Byzantium imperial city Constantinople under Ottoman rule 1453

The birth of Greek mathematics went hand in hand with the rise of rational speculation on the nature of reality, a new way to think free from earlier myths and based on logic and argument. It all started in the 6^{th} century BC on the Ionian coast of Anatolia in modern Turkey. Why the origin of scientific thought happened there and then is a matter of speculation but we know that early Greek philosophers travelled widely and picked up knowledge of mathematics and astronomy from Egypt, Babylonia and Persia. A new idea started to form, mathematics for its own sake, a formal approach that used axioms, logical arguments and proofs to secure knowledge of an idealized world of abstract concepts.

Thales of Miletus (c. 624 – c. 546)

Greek philosophy starts with Thales from Miletus on the western coast of Anatolia. He explained natural phenomena from natural causes with no gods involved. The study of nature became known as “Natural philosophy”, a term that lasted two millennia.

We know of Thales not from any surviving texts written by him since there is none. Books that mention him and his works include Aritotle’s *Metaphysics* and Diogenes Laërtius’ *Lives and opinions of eminent philosophers.*

As many Greek philosophers Thales sought a basic element behind the bewildering complexity we see around us. A bold idea that has been very fruitful to understand the world, just think of atoms, elementary particles, unified theories and possibly strings. Thales argued for water as the primary substance. Pre-Socratic philosophers followed his lead with their own theories of fundamental substances. Empedocles (490–430 BC) suggested four elements: fire, air, water, and earth which became a central dogma for two millennia, sometimes complemented by a fifth element, the aether. Similar ideas with five basic elements occur in Hinduism and Buddhism.

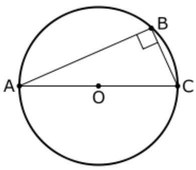
Thales made contributions in many fields: philosophy, history, astronomy, science, mathematics, engineering, geography and politics. He is known as:

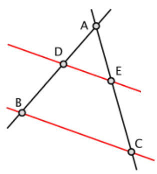
- Initiator of the Ionian enlightenment
- Founder of the Milesian school of natural philosophy
- Chief of the seven sages, philosophers and statesmen of the 6^{th} century BC
- Father of science
- Founder of abstract geometry
- The first philosopher

A renaissance man 2000 years before the Renaissance.

What did Thales do to deserve all this praise? Some examples of things attributed to Thales are:

- Astronomy
  The first successful prediction of a solar eclipse that happened in 585 BC.
  Suggesting Ursa Minor and the Polar star for finding true north,
  a practice used by Phoenician sailors
- Physics and Geology
  Describing static electricity and magnetism but not a pure materialist, matter had a ‘soul’, a life-giving force causing motion and change.
  Believed in a spherical earth. The ground floated on water which could explain earthquakes.
- Mathematics
  Travelled to Egypt and studied geometry. Proved Thales’ theorems:
  A circle with diameter AC and B a point on the perimeter has $\angle ABC = 90°$
  Ratios of corresponding sides in similar triangles are equal. He used this to calculate the heights of pyramids and the distance to ships off the shore.





$\Delta ADE \sim \Delta ABC$

$$\frac{DE}{BC} = \frac{AE}{AC} = \frac{AD}{AB}$$

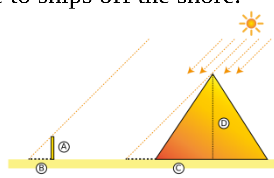


Fig 2.9.2 Accomplishments of Thales.

Mathematical and philosophical development in ancient Greece was often associated with a specific place and a leader with a new idea that attracted pupils. Different centers with educations, scholars and a learned debate formed and succeeded each other. The first of these schools were the Ionian school formed by Thales from Miletus. His most celebrated followers are Anaximander (c.610–c.546) and Anaximenes (c.585–c.528). They believed nature was ruled by laws that could be used for predictions. Their theories are known from some surviving fragments of their writings and from later commentaries. Theirs was the first rational view of nature without gods or other mythical beings. It was mostly based on bold speculations and guesses rather than systematic observation. The school declined after the Persian invasion of the region.

Pythagoras (c. 570 – c. 495 BC)

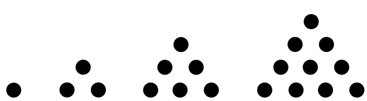
Pythagoras has been described as the first true mathematician but the society he founded was more about religion and philosophy than science. The Pythagoreans followed a code of secrecy and early writers with access to original sources where more devoted to building a myth than giving a true picture. Our knowledge of Pythagoras is full of uncertainties. Many mathematical achievements attributed to him are most probably the work of his followers. Pythagoras' theorem had been known for a thousand years and was proved by Apastambha in India before the Pythagoreans.

Pythagoras was born on Samos near the Turkish coast. He probably met Thales and attended lectures on geometry and astronomy given by Anaximander in Miletus. Many of his religious ideas seem to have come from his visit in Egypt. During the wars with Persia he became a prisoner of war and was taken to Babylon where he would learn about arithmetic, music and Mesopotamian religion.

After returning to Samos and taking part in politics he would later move to Croton, a Greek colony in southern Italy where he founded a philosophical and religious school. His teachings were inspired by mathematics, music, astronomy and religion. The society had strict rules on behavior and diet; today we would probably call them ascetic mystics. One of their beliefs was that reality at its deepest level is mathematical in nature. The inner circle was known as mathematikoi. Both men and women were allowed as members and many of their famous philosophers were women which was very unusual at the time

Pythagoras interest in mathematics was focused on its philosophy. What is the relation between numbers or geometrical figures and reality? What does it mean to prove something? We are used to mathematical abstractions but in those days these were new concepts and some would say that they were first formulated by the Pythagoreans.

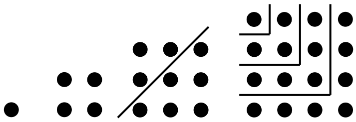
Pythagoras who played the lyre discovered that the sounds of vibrating strings combine harmonically when the lengths of the strings are in certain integer ratios. According to Aristotle the Pythagoreans regarded whole numbers as the building block of material objects, they did not exist free from sensory objects.[15] Pythagoreans pictured whole numbers as geometrical dots, each number had its own personality, masculine or feminine, perfect or incomplete etc. Numbers were classified by geometrical arrangements of dots.

Triangular numbers:  …

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Fig 2.9.3 Pythagorean arrangement of integers

Four was a favorite number but ten was seen as the most interesting number. It was triangular and consisted of the first four numbers 1+2+3+4.

Square numbers:  … n^2

By doing geometrical arrangements of numbers they would realize certain algebraic identities:

The sum of two consecutive triangular numbers is a square number:

$$\frac{n(n+1)}{2}+\frac{(n+1)(n+2)}{2}=(n+1)^2$$

Add a ⅃-shape to get the next square:

$$n^2+(2n+1)=(n+1)^2$$

Decompose a square into ⅃-parts

$$1+3+5+\cdots+(2n-1)=n^2$$

This is the origin of our use of geometrical language for algebraic concepts like the square for x^2 and the cube for x^3. Composite numbers that were not square were called oblong. Prime numbers are non-composite numbers, they were recognized as a special group to study.

Prime numbers have deep and interesting properties, they will play an important part in the analysis of our problems from chapter one but periodic fractions were never an issue for Greek mathematicians; their interest was focused on geometry. Numerals were represented additively based on powers of 10. Decimals were introduced much later by Hellenistic astronomers that used the Babylonian sexagesimal system based on powers of 1/60.

In Greek mathematics x^2 (if they had used this notation) was never thought of as a number multiplied by itself but as a square constructed on a segment of length x. Pythagoras' theorem for a right-angled theorem meant that the two squares a^2 and b^2 could be cut up and reassembled to a square c^2.

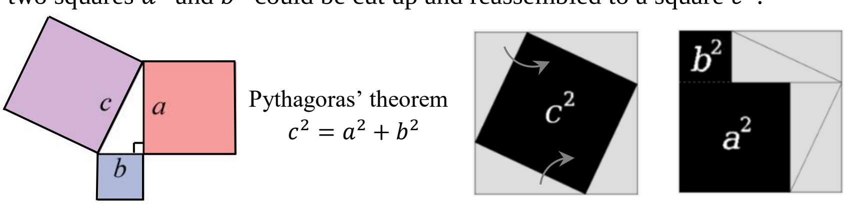


Fig 2.9.4 Pythagoras' theorem and a visual proof.

For Pythagoreans a number was a positive integer. Ratios of lengths in a geometrical figure were considered but they were not seen as numbers. Fractions occurred in commerce and practical application but they were not part of mathematics. A ratio with two integers was called commensurable.

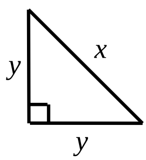
The discovery of incommensurable ratios called inexpressible must have been very disturbing to Pythagoreans with their belief in a universe built from whole numbers. If two lengths were incommensurable there could be no common unit that made each length a multiple of this unit, no matter how small it was. Legend has it that Hippasus made the discovery on sea and that he was subsequently thrown overboard by his Pythagorean "friends".

Hippasus' proof of incommensurable ratios

by *reductio ad absurdum*

Assumption ⇒ Contradiction

$\sqrt{2}$ cannot be expressed as a fraction of two integers



$$x^2 = y^2 + y^2$$
$$x/y = \sqrt{2}$$

Assume $\sqrt{2} = \frac{\alpha}{\beta}$ with $\alpha, \beta \in \mathbb{N}_1$ and $(\alpha, \beta) = 1$ (no common factor)

$\frac{\alpha^2}{\beta^2} = 2 \;\Rightarrow\; \alpha^2 = 2\beta^2 \;\Rightarrow\; \alpha = 2m, m \in \mathbb{N}_1 \;\Rightarrow\; 2m^2 = \beta^2 \Rightarrow$

$\beta = 2n, n \in \mathbb{N}_1 \;\Rightarrow (\alpha, \beta) \geq 2 \;\Rightarrow$ Contradiction

$\sqrt{2}$ must be an irrational number, x and y are incommensurable lengths.

Pythagorean ideas influenced Plato and Aristotle and thereby all of western philosophy (A historical concept since true philosophy should be as universal as mathematics). In the philosophy of mathematics there are two viewpoints; mathematical abstractions as human inventions or discoveries. Mathematical theorems seen as discoveries in a fixed world of ideas can be traced back to Plato's *theory of Forms* with roots in Pythagorean thinking. Finally, here is a quote from *A History of Western Philosophy* written by Bertrand Russel: [Pyhtagoras] *was intellectually one of the most important men that ever lived, both when he was wise and when he was unwise. Mathematics, in the sense of demonstrative deductive argument, begins with him.* [16]

Zeno, Hippocrates and Eudoxus

These three philosophers/mathematicians were all involved with the question of continuity and limits. The discovery of incommensurable ratios illustrates a problem that occupied many Greek thinkers; the relation of the discrete to the continuous. There could be no common measure that made each length a multiple of this unit, no matter how small the unit measure. One of these thinkers was **Zeno** of Elea (c.490–c.430 BC) in southern Italy. He was a member of the Eleatic school founded by Parmenides, the philosopher known for claiming that change is impossible. The opposite was claimed by Heraclitus who said that everything changes "Panta rhei" (No man ever steps in the same river twice).

There were two opposing views of space and time. One held that space and time are infinitely divisible and motion could be smooth and continuous. The other view claimed that space and time are made up of small indivisible intervals like frames in a movie. Zeno argued with many paradoxes against both theories. The first paradox claims that motion is impossible; to cover a distance you must first cover half the distance then a quarter of the distance and so on. You would have to cover an infinite number of distances in a finite time. The second paradox called Achilles and the Tortoise argues in a similar way that a fast object can never catch up with a slow object. My own version of how to think on these matters is a lamp turned on for ½ second then turned off for ¼ second etc. Will the lamp be on or off after one second?

A mathematician and a physicist would solve this differently. The mathematician might introduce a function $f(t)$ with 1/0 for the lamp being on/off.

$$f(t) = \begin{cases} 1 \text{ if } t \in \left[\frac{0}{2}, \frac{1}{2}\right[\cup \left[\frac{6}{8}, \frac{7}{8}\right[\cup \left[\frac{30}{32}, \frac{31}{32}\right[\cup \dots \\ 0 \text{ if } t \in \left[\frac{2}{4}, \frac{3}{4}\right[\cup \left[\frac{14}{16}, \frac{15}{16}\right[\cup \left[\frac{62}{64}, \frac{63}{64}\right[\cup \dots \end{cases}$$

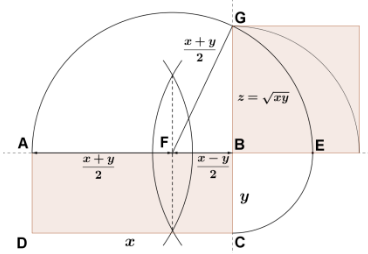
What is $f(1)$?

Answer: The function is only defined for $0 \leq t < 1$. We could enlarge the domain of definition and define $f(1)$ as either 1 or 0 without contradiction. A physicist might say that no mechanism could take an arbitrary number of switches. It would have to break some time and stay off a $t = 1$ or he could say that time intervals shorter than the Planck scale $\Delta t = 5 \cdot 10^{-44}\ s$ are not well-defined, the question must be rephrased to be meaningful.

Zeno may have been the first philosopher in the Greek tradition to deal with mathematical infinity and limits.

The need to think of limits would reappear in the discussion of areas. The area of a figure was decided by constructing a square with the same area. Only ruler and compass were allowed. The process was called quadrature. To find the quadrature of the circle became known as "squaring the circle". To solve it for a unit circle you need to construct a side of length $\sqrt{\pi}$ which has been proved to be impossible.

Squaring a rectangle is done as follows:



Extend AB and put E so that BC=BE.
Bisect AE and draw a semicircle on AE.
Extend BC to an intersection at point G.
From BG construct a square.
If rectangle ABCD has sides x and y
then BG will be of length $\sqrt{xy}$.

Fig 2.9.5 Geometrical construction with ruler and compass.

This construction produces the geometrical mean of x and y, $G = \sqrt{xy}$ With the property that x, G, y are in a geometrical progression $G/x = y/G$. The arithmetical mean $A = (x + y)/2$ makes an arithmetical progression x, A, y with $A - x = y - A$. There is a third mean called harmonic mean H defined by $H^{-1} = (x^{-1} + y^{-1})/2$. Collectively they are known as the Pythagorean means. If x and y are positive: $\min(x, y) \leq H \leq G \leq A \leq \max(x, y)$ with equalities iff (if and only if) $x = y$. A fourth and very useful mean is the quadratic mean Q aka root mean square (rms) defined by $Q^2 = (x^2 + y^2)/2$.

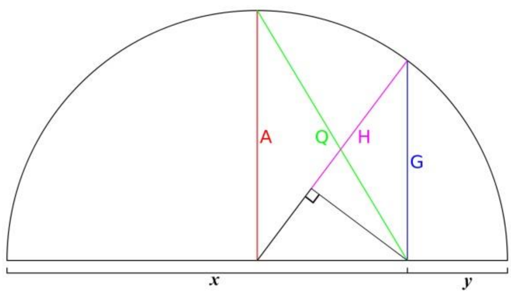


Fig 2.9.6 Arithmetical, Quadratic, Harmonic and Geometrical mean

Area is additive for non-overlapping figures and invariant under translation and rotation. This gives the area of a triangle. The area of a figure bounded by straight edges, a polygon can be found by triangulation. The area of a rounded object is found as a limit of inscribed polygons that approach the desired area (method of exhaustion).

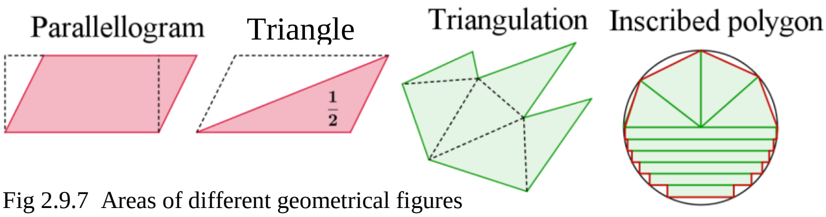


Fig 2.9.7 Areas of different geometrical figures

Volumes are treated in a similar fashion with "triangulation" based on the simplest 3-dimensional polyhedron, the tetrahedron. Triangulation in n-dimensional space is based on the n-simplex, a polytope consisting of the convex hull of $n+1$ points.

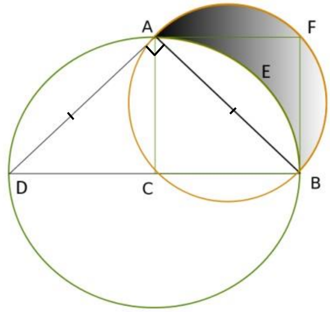


Fig 2.9.8 Lune of Hippocrates

Hippocrates of Chios (c.470–c.410 BC) was the first to make progress on calculating a rounded area. He assumed that ratios of semicircular areas equaled the squared ratios of their diameters:

$$\frac{\text{Big semicircle area}}{\text{Small semicicrcle area}} = \frac{\text{BD}^2}{\text{AB}^2} = \frac{\text{AB}^2 + AD^2}{AB^2} = 2$$

Remove the area ABE common to the big quarter circle and the small semicircle and the area of the shaded lunar area equals the area of △ABC. The area of a curvilinear figure has been reduced to a rectilinear figure. A proper proof of Hippocrates assumption depends on the method of exhaustion which was unknown to him. It was introduced by **Eudoxus** of Cnidus (408–355 BC).

Ancient Greek history has two parts, the classical period (c.500–323 BC) and the Hellenistic period (323–146 BC). The Hellenistic period begins with the death of Alexander III and ends with the Roman conquest. Eudoxus is considered the greatest of Greek mathematicians of the classical period. The greatest of all times is Archimedes from the Hellenistic period. Eudoxus was born in Asia Minor and was quite poor but made extensive travels and got education from the best scholars of his time. He studied astronomy in Egypt, mathematics from Archytas, a Pythagorean in Italy, medicin from Philiston in Sicily and philosophy from Plato in Athens. It is said that he could not afford an apartment in Athens and had to walk from Pireaus 11 km from Athens each day to attend Plato's lectures.

Eudoxus incorporated the irrational numbers into mathematics. The Pythagoreans that had discovered them avoided them and treated them as numbers not to be counted with.

Numbers and arithmetic had been fundamental to earlier Greek mathematicians but Eudoxus introduced a sharp distinction between numbers and geometry. He introduced a notion called magnitude for geometrical quantities such as length, angle, area and volume. Magnitudes could vary continuously in contrast to numbers that were discrete. Irrational quantities were incorporated into a logical foundation of geometry but at the same time they were isolated from numbers and calculation. After Eudoxus classical Greeks abandoned algebra and focused on geometry which became the basis for mathematics, a tradition that had serious effects on European mathematics for two millennia.

On the positive side Eudoxus began the development of infinitesimal calculus with the method of exhaustion to calculate areas, he viewed π as the limit of polygonal perimeters, he discovered the formula for the sum of a geometric series, started with solid geometry and calculated the volume of a cone and he established the tradition of deduction based on explicit axioms.

Eudoxus introduced the Archimedean property to handle Zeno's paradoxes. As is often the case with mathematical naming; historical circumstances rather than being the first to discover something will decide the name of a mathematical concept or a theorem. Roughly speaking a number system has the Archimedean property if there are no infinitely large or infinitely small elements. Let x and y be positive numbers then x is infinitesimal with respect to y if $nx < y$ for every $n \in \mathbb{Z}$. If x is infinitesimal then $1/x$ is larger than any given integer. The real numbers can be expanded into larger systems that are non-Archimedean. Examples are hyperreal numbers and surreal numbers. The number $0.999\ldots$ from chapter one is not infinitesimally close to 1 since they are identical, if ε were a positive infinitesimal then $0.999\ldots + \varepsilon > 1$.

Eudoxus was also a great mathematical astronomer with his own theory of planetary orbits and he has been quoted as saying "Willingly would I burn to death like Phaeton, were this the price for reaching the sun and learning its shape, its size and its substance."

Sophists, Socrates, Plato and Aristotle

Persian defeat in the battle of Mycale (479 BC) and aversion to Spartan leadership resulted in a new anti-Persian alliance under Athenian leadership, the Delian League. Athens grew rich and became an intellectual center under the leadership of Pericles. Parthenon and other glorifying temples were built on the Athenian acropolis. Pythagoreans and Sophists that taught philosophy, ethics, rhetoric, geometry and astronomy were attracted to Athens.

Fig 2.9.9 Academy of Athens, with Athena-Apollo and Plato-Socrates.

Many mathematical achievements of the Sophists were a result of their search for solutions to the non-solvable geometrical construction problems; squaring the circle, constructing a cube of double volume and trisecting an angle.

Socrates (470–399 BC) was a teacher like the sophists but he criticized them for only giving wisdom to those who could pay for it. The term sophism comes from σοφός, sophós which means ”wise man”. Philosophy comes from φιλοσοφία, philosophia “love of wisdom” a term that may have been coined by Pythagoras. Socrates method of teaching was to dismantle a problem into a series of question that would lead the student to an answer, an early precursor to the scientific method.

There are no extant texts written by Socrates. Our best knowledge about him comes from his contemporaries; Xenophon the historian, Aristophanes the comedy playwright and Plato his disciple who wrote about him in *Plato's dialogues*. The picture that appears is somewhat contradictory but there is agreement that he had a brilliant intellect and an ugly appearance.

Socrates differed from the Pre-Socratic philosophers, a group that strived for rational explanations of natural phenomena without myths or deities. Socrates' disciple Aristotle called them *physikoi* from the word for nature *physis*. Among them were Leucippus and his pupil Democritus, a contemporary of Socrates. They believed that everything was composed of indivisible units called atoms. Socrates main interest was ethics. A saying attributed to Socrates is “I only know that I know nothing”.

Socrates clashed with Athenian politicians and was accused of seducing the youth with his teachings of justice and ethics. He was sentenced to death and met the death calmly with a drink containing poison hemlock.

Socrates most famous follower was **Plato** (c.425–347 BC). He came from a wealthy family involved in politics. Plato too had political ambitions but the fate of Socrates convinced him that there was no place in politics for a man of conscience. On political matters Plato favored a utopian society ruled by philosophers. In later works his ideal was more totalitarian like Sparta, not like the democratic Athenian state that had condemned Socrates to death.

Most works of predecessors and contemporaries of Plato are forever lost but all of Plato's works have survived. They are written in the form of dialogues where historical persons represent different viewpoints. In the dialogue *Apology* dealing with Socrates' defense speech Plato describes himself as a devoted young follower of Socrates.

Plato travelled to Egypt and Italy where he learnt mathematics from Pythagoreans. Their influence can be seen in Plato's theory of Forms with a sharp distinction between a world of ideas and a world of things. This is illustrated in his allegory of the cave. The physical world was imperfect and subject to change and decay. Only the ideal world is worthy of study to provide certain knowledge. On the physical world we can have only opinions based on our fallible senses.

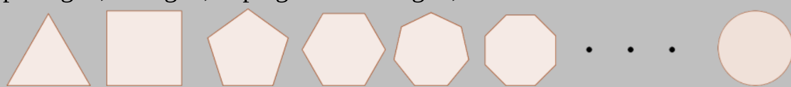
Plato did not do much mathematics himself but he regarded it as a vital subject for philosophy and for the understanding of the universe. Mathematical concepts were seen as independent of the material world with a reality of their own, they were discovered rather than invented. Many modern mathematicians share his view but not all. Plato wanted a deductive system of knowledge; mathematics should be based on rigorous proofs and arrangement of theorems in a logical order. This opened the question of whether a certain problem could be solved based on given assumptions or axioms.

Plato's works focused on ideas: truth and goodness, the ideal society and the perfect state, the nature of matter and mathematics. To understand the focus on philosophy and abstractions in Greek society you need to understand the society. Classical Greek society had two distinct classes; an upper class of citizens known as free men. They had education, right to vote, leisure time and some contempt for the lower class of slaves that were involved with respectable professions like commerce and medicin. Plato and his student Aristotle said that no citizen should be involved with practical applications. These attitudes were very different from the time when Thales lived, a time when work, trade and technical skills were honorable and mathematics was seen as a tool for practical applications.

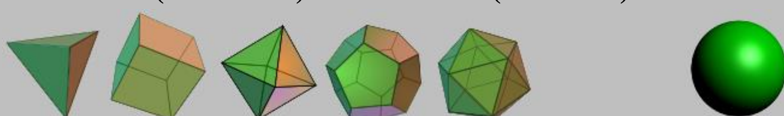
Plato taught at the academy of Athens which he founded in 387 BC. It was like a small university with studies in mathematics and philosophy.

Polygons and polyhedra, from lower to higher dimensions

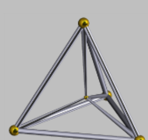
Our starting point is points, 0-dimensional objects called 0-polytopes, followed by 1-polytopes, 1-dimensional objects bounded by 0-polytopes, line segments. All points within a given distance from a center form a 1-ball. The ball packing problem described on page 2 becomes trivial, the density is 100%. For two dimensions, 2-polytopes or polygons are areas bounded by 1-polytopes. Below are: equilateral triangle, square, regular pentagon, hexagon, heptagon and octagon; named with Greek numerals.



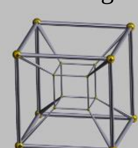
The limit of regular convex polygons is a disc, a 2-ball that is bounded by a circle, a 1-sphere. The densest packing of circles may seem obvious, a regular hexagonal pattern. It was proved in 1773 by Lagrange for regular cases but a rigorous proof for all packings came first 1940 by Lászlo Tóth who also showed that the Kepler conjecture could be reduced to a finite case analysis solvable by a computer. For three dimensions we get polyhedra or 3-polytopes, volumes bounded by 2-polytopes. The regular ones have congruent, regular polygons as boundaries, the same number at each vertex. They are known as Platonic solids: tetrahedron, cube, octagon, dodecahedron (dodeka = 12) and icosahedron (eikosi = 20).



Theaetetus, a contemporary of Plato showed there were no other regular polyhedrons. The cube and the octagon are duals of each other since the vertices of one form the face-centers of the other. The same goes for the last two whereas the tetrahedron is self-dual. Plato associated each of the basic elements with one Platonic solid based on some odd arguments. Aristotle added a fifth element without fulfilling the list of Plato. In four dimensions we get 4-polytopes bounded by 3-polytopes. There are six regular convex 4-polytopes, two dual pairs and two self-duals, the new one is the 24-cell. For five dimensions and up there are three regular ones.



Projected wireframe models of
4D - tetrahedron: 5-cell
4D - cube: tesseract



After Alexandrian conquests and the spread of Greek culture the center of mathematical studies shifted to Alexandria but the academy remained the center of philosophy. The academy was destroyed in 84 BC by Sulla, Roman general and statesman who attacked and occupied Athens. Neoplatonists revived the academy in 410 AD until it was closed by the Christian emperor Justinian in 529 AD for teaching “pagan and perverse learning”.

That did not stop Plato’s influence on coming generations, a quote from the Stanford Encyclopedia of Philosophy: “the subject of philosophy, as it is often conceived - a rigorous and systematic examination of ethical, political, metaphysical, and epistemological issues, armed with a distinctive method - can be called his invention. Few other authors in the history of Western philosophy approximate him in depth and range”.

Aristotle (384–322 BC) grew up close to the Macedonian monarchy. His father was the personal physician to the king. When he was 18 he joined the Academy to become a pupil of Plato and remained there for 20 years.

Many but not all of Aristotle’s texts have been preserved. He was interested and produced work in every field that could interest an educated man from classical Greece: logic, metaphysics, astronomy, mechanics, biology, botany, zoology, ethics, aesthetics, linguistics, rhetoric, poetry, theater, music, geography, politics, government and more. He left the Academy when Philip II, king of Macedon wanted him to tutor his son Alexander in 343 BC.

Aristotle and Plato had different theories of ontology and epistemology. Plato believed that true knowledge started from ideas. Aristotle believed that all knowledge and concepts were based in perception. Numbers and geometrical forms were properties of real objects just as properties like hardness and warmth were part of objects. The difference is illustrated in Raphael’s fresco where Plato points up to a world of ideas that can be reached by the intellect and Aristotle gestures down to a world of objects to be studied by our senses.

fig 2.9.10 School of Athens, Plato and Aristotle surrounded by historical figures.

Aristole’s work most relevant for mathematics is his study of logic that is described in six works named *the Organon*. It deals with syllogisms that in their simples form goes like: All men are mortal, Socrates is a man therefore Socrates is mortal. He was not the first to study sound deductions. The Stoic school of logic studied propositions, compositions and conditional statements. History of logic does not start with Greek culture; India had produced several schools of logic centuries before. Panini used sophisticated logical rules in his studies of Sanskrit grammar and the Mohist school in China used strict logic to formulate legal rules. Modern logic replaces these old traditions with predicate logic in the late 19^{th} century.

Aristotle’s method for studying empirical subjects was different from the modern scientific method with competing explanations and a winner decided by experiment or observation. It was more speculative relying on qualitative observations. As a consequence most of his theories were fundamentally flawed especially in physics where he lacked fundamental quantitative concepts like mass and velocity needed for real progress. Aristotle believed in the four basic elements of Empedocles; fire, air, water and earth but then he added a fifth element. On earth everything changed but the heavens were constant. A new and unchangeable substance was needed for the heavens and the stars. Aristotle called it aether.

Greek thinking was lost for a long period in Europe (c.600–c.1100) when intellectual life was replaced by Christian theology. On its return Aristotelian theories were adopted by the church and made into fixed doctrines. It would take five hundred years before they were challenged by modern science with experiments and observations based on careful, quantitative measurements and scientists like Brahe and Galilei. “Almost every serious intellectual advance has had to begin with an attack on some Aristotelian doctrine”.[17]

Most of Aristotle’s theories were a big step forward. Politically he differed from Plato. He advocated rule with moderation and no excesses. Aristotle can hardly be blamed for how many of his ideas were made into unquestionable facts by later generations with a total lack of intellectual curiosity something that Aristotle, safe to say, did not lack.

Pericles had united the Greek city-states and formed a coalition that drove the Persians out of the Aegean. The Athenian empire was in conflict with the Peloponnesian league led by Sparta. For a generation they fought in the Peloponnesian war (431 – 404). The Greek states were exhausted after much war and internal conflict. Philip II, king of Macedon in the north took advantage of the situation and invaded the Greeks. Philip united Greeks and Macedonians against Persians who ruled over Greeks in Asia Minor. He was assassinated but his young son Alexander, pupil of Aristotle carried on his mission to liberate Greeks from Persian control. Alexander was victorious on the battlefield and became the ruler of an enormous empire.

Greek cultural and mathematical history enters a new era called Hellenistic where Greek culture dominates in a vast empire stretching over Egyptian, Persian and Babylonian civilizations. Alexander dies young and his empire is divided into several parts. The vast eastern part forms the Seleucid Empire ruled by the dynasty of Seleucid and Egypt becomes a kingdom under the Ptolemaic dynasty. Its center Alexandria becomes the global center of knowledge in fields like astronomy and mathematics.

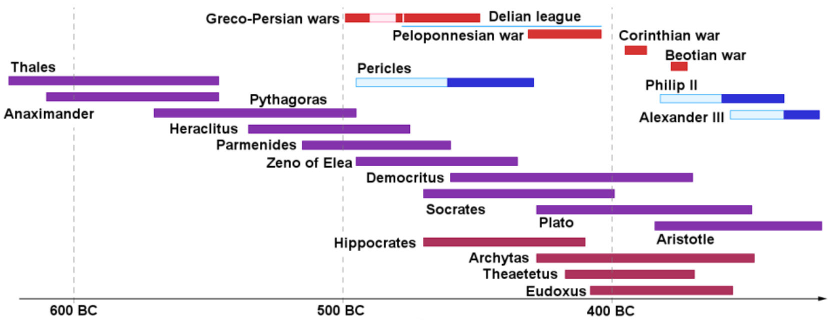


Fig 2.9.11 Wars, rulers, philosophers and mathematicians in classical Greece.

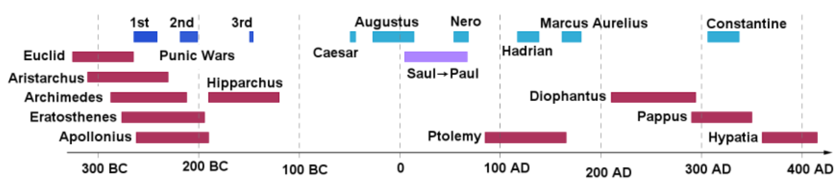


Fig 2.9.12 Timeline of Hellenistic and Roman era.

Very little is known of **Euclid** the man who wrote a textbook called *The Elements of Geometry* commonly known as Euclid's *Elements*. It is the most influential mathematical work and it was the main mathematical textbook for more than 2000 years. Euclid, known as "The father of geometry" was active in Egypt during the reign of Ptolemy I (323–283 BC) after the Alexandrian conquest. He was a leading mathematician in Alexandria, the new center of culture and knowledge with its world famous library that belonged to the "Musaeum of Alexandria" ("Institution of the Muses", the nine goddesses that inspired literature science and the arts).

Euclid and his book represent a transition from the classical Greek period and the following Hellenic period. The book is a summary of mathematical knowledge up to Euclid's time, from Pythagoras to theorems of Theaetetus and Eudoxus. Mathematics is presented in a systematic and deductive fashion with definitions, axioms and theorems. The theorems have rigorous proofs and are used as building blocks (elements) for later theorems. The content of the *Elements* is divided into 13 books.

Book I has 23 definitions in the form of intuitive explanations of concepts like points, curves, straight lines, parallel lines, circles, planes and surfaces.

1. A point is that which has no part.
2. A curve is a length without breadth.
3. The extremities of a curve are points

⋮

23. Parallel lines do not meet when extended indefinitely in either direction.

Then there are five axioms called common notions that are valid generally.

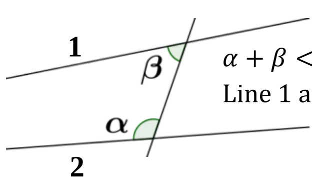
1. Things equal to the same thing are equal to one another
2. If equals are added to equals, the results are equal

⋮

5. The whole is greater than the part.

And finally five axioms of geometry called postulates.

1. It is possible to draw a straight line between any two points.
2. It is possible to extend a finite straight line continuously in a straight line.
3. It is possible to construct a circle with any center and any radius.
4. All right angles are equal to one another.
5.



$\alpha + \beta < 90^\circ \implies$
Line 1 and Line 2 will meet at the side of the angles.

Euclid's great legacy is his inclusion of postulate five known as the parallel postulate. It does not seem as self-evident as the others and many objected to it and tried to prove it from the other postulates. Euclid did treat it as special and whenever possible he proved theorems without using the fifth postulate. There are many alternatives to postulate five that give the same theorems. One alternative is "the sum of angles in every triangle is 180°".

There can be no proof of the parallel postulate or its alternatives from the other postulates. This can be shown by presenting a consistent geometrical model where postulate 1–4 are true but the fifth postulate is false. This was shown in the 19th century by Gauss, Bolyai and Lobachevsky. Alternative geometries are called non-Euclidean. The simplest case is the spherical geometry where straight lines based on shortest distance between points coincide with great circles.

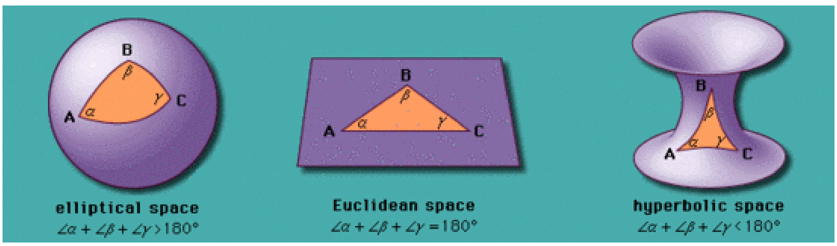
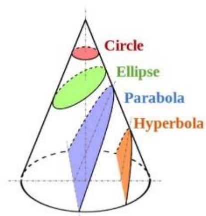


Fig 2.9.13 Euclidean and non-Euclidean geometries

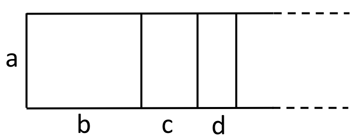
The sum of angles in a triangle decides the kind of geometry, elliptic if above 180° and hyperbolic if below 180°. These names together with parabolic recur over and over again in mathematics. Euclid used them in geometrical constructions that solved quadratic equations. *Elleipsis* means falling short or deficiency, *parabole* means setting side-by-side or comparable and *hyperbole* means setting above or excess. The same terms are used for conic sections depending on if the inclination of the cutting plane is below, equal or larger than the side of the cone. What postulates should be used?

Aristotle's criterion was that correct postulates should lead to theorems that agreed with reality. Later Greek philosophers like Proclus (~450 AD) from the Neoplatonist school would consider all of mathematics as hypothetical, deducing consequences from chosen assumptions.

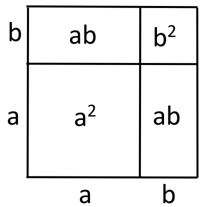


The theorems in book I are statements about rectilinear figures and circles. Geometrical constructions, angles, triangles, parallelograms and areas are treated. Pythagoras theorem is dealt with in proposition 47.

Book II contains basic school algebra but in a geometrical setting, there were no numbers spread out on a number line with negative and irrational values. What they had instead was line segments. Addition and subtraction was handled by extending and reducing line segments. Multiplication was done by areas and volumes. No multiplication contained more than three factors since there were no objects of higher dimension. Geometrical problems that we would solve with variables and equations were handled with geometrical construction. Geometrical algebra:



$a(b + c + d + \cdots) = ab + ac + ad + \cdots$



$(a + b)^2 = a^2 + 2ab + b^2$

Book III handles circles and theorems familiar from high school geometry.

Book IV deals with figures inscribed in circles, polygons circumscribed by circles and construction of regular polygons. It ends with the construction of a regular 15-sided polygon. The question of which regular polygons that can be constructed with ruler and compass will turn out to be quite an interesting question. It was solved in 1796 by Carl Friedrich Gauss when he was just 19. He was so pleased with his discovery and his construction of a regular 17-gon that he decided to become a mathematician instead of a philologist. It was his wish that a regular heptadecagon should be inscribed on his tombstone. The first non-constructible regular polygon is the regular heptagon. A nice way to form a regular pentagon is to make a knot on a paper ribbon and flattening it.

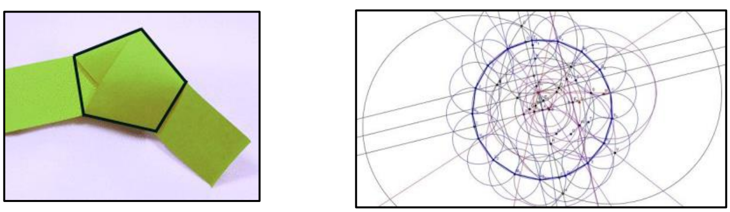


Fig 2.9.14 Construction of pentagon and heptadecagon

Book V is about division. Magnitudes and their ratios are introduced, commensurable or incommensurable. Magnitudes are used for things like length, angle, weight and time. Only magnitudes of the same kind can form a ratio. They look like numbers but they are not. They are never added or multiplied only compared. A complex definition is given to compare the size of two ratios a/b and c/d.

Book VI use ratios to handle similar figures, areas and geometrical problems corresponding to quadratic equations.

Books VII, **VIII** and **IX** cover arithmetic and number theory, reaching back to the Pythagorean tradition. The concepts of divisors and prime numbers are defined and a method for finding the greatest common divisor (GCD) is presented. It is shown that the factorization of integers into prime numbers is unique. This is not always true, $\mathbb{Z}\left[\sqrt{-5}\right] = \left\{a + b\sqrt{-5} \middle| a, b \in \mathbb{Z}\right\}$ is a number system with $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. Elegant proofs for the sum of a geometric progression and for the infinite number of primes are given.

Euclid assumes there are only a finite number of primes $p_1, p_2, \ldots, p_n$ and forms a new number $P = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$ that is either a new prime not in the list or factors into primes not in the list since none of p_i divides P, this contradicts the assumption of a finite number of n primes.

Book X classifies different types of incommensurable lines, numbers that in modern algebra would be written as $\sqrt{\sqrt{a} \pm \sqrt{b}}$ with a and b commensurable.

Books XI, **XII** and **XIII** contain solid geometry, the method of exhaustion and a rigorous limit procedure to calculate areas of curvilinear figures. Important proofs are the volume of a cone as a third of a cylinder of the same base/height and that the ratio of the volumes of two spheres is the same as the cube of their diameters ($V'/V = (r'/r)^3$). The actual formula for the volume of a sphere had to wait for Archimedes. Book XIII contains the constructions and inscriptions in a sphere of the five regular convex solids. In total the thirteen books contain 467 propositions.

Euclid's Elements became the most enduring of all textbooks with over thousand different editions. It was required reading for students well into the 20th century. One edition was made in the 4th century by Theon of Alexandria, mathematician and father of Hypatia. His version became the basis for all editions until the early 19th century when an older version was found in the Vatican Library.

There are four or five other surviving works of Euclid, all written in the same deductive style with definitions and proved propositions. *Phaenomena* deals with spherical astronomy, *Catoptrics* treats the theory of mirrors and *Optics* deals with perspective and vision. Euclid also wrote a book on conic sections that did not survive but is believed to form the first four books of Apollonius' very influential work *Conics*.

Euclidean algorithm and continued fractions

The greatest common divisor of two or more integers is a very useful concept. For example $\mathrm{GCD}(30{,}42) \equiv (30{,}42) = 6$. It can be calculated by factorization $\left(\prod_{i=1}^{\infty} p_i^{a_i}, \prod_{i=1}^{\infty} p_i^{b_i}\right) = \prod_{i=1}^{\infty} p_i^{\min(a_i,b_i)}$. The Euclidean algorithm described in the *Elements* does this more effectively. It starts from division of two integers a/b with quotient (q) and remainder (r).

$$\frac{a}{b} = q + \frac{r}{b} \quad q \in \{0,1,\dots\} \quad r \in \{0,1,\dots,b-1\} \qquad a = qb + r$$

$$(a,b) = (qb + r, b) = (r,b) \qquad \text{Iterate this procedure}$$

$$\begin{array}{ccc}
a = q_0 b + r_0 & r_i \geq 0 & (a,b) = \\
b = q_1 r_0 + r_1 & b > r_0 > r_1 > \cdots & (b,r_0) = \\
r_0 = q_2 r_1 + r_2 & \Downarrow & (r_0,r_1) = \\
\vdots & \exists n{:}\, r_n = 0 & \vdots \\
r_{n-2} = q_n r_{n-1} + 0 & & (r_{n-2},r_{n-1}) = r_{n-1}
\end{array}$$

If $(a,b) = 1$ then a and b are called relatively prime or coprime.

The Euclidean algorithm leads to a representation of numbers that is more fundamental than decimal representation since it is independent of base:

$$\begin{aligned}
\frac{a}{b} &= q_0 + \frac{r_0}{q_1 r_0 + r_1} = q_0 + \frac{1}{q_1 + r_1/r_0} \\
&= q_0 + \frac{1}{q_1 + \frac{1}{q_2 + r_2/r_1}} = \cdots \\
&= q_0 + \frac{1}{q_1 +}\,\frac{1}{q_2 +}\cdots\frac{1}{q_{n-1} +}\,\frac{1}{q_n} \quad \text{(Continued fraction)} \\
&\equiv [q_0; q_1, q_2, \cdots, q_n] = [\,q_0; q_1, q_2, \cdots, q_{n-1}, q_n - 1, 1]
\end{aligned}$$

x irrational number $\Leftrightarrow\ x = \lim_{k\to\infty} [a_0; a_1, a_2, \dots, a_k] \equiv [a_0; a_1, a_2, \dots]$

$a_k = \lfloor x_k \rfloor$: $x_0 = x$ and $x_{k+1} = 1/(x_k - a_k)$ $\qquad (\, a_0 \in \mathbb{Z}\,, a_{i>0} \in \mathbb{Z}^+\,)$

$\sqrt{19} = [4; \overline{2,1,3,1,2,8}\,]$	Periodic representation $\Leftrightarrow x = (a + b\sqrt{c})/d$
$e = [2; 1,2,1,1,4,1,1,6, \dots]$	Infinite pattern $\Rightarrow$ e is an irrational number
$\pi = [3; 7,15,1,292,1, \dots]$	Seemingly random numbers
$\varphi = [1; 1,1, \dots]$	Golden ratio, "the most irrational number"

Proofs and more details are given in appendix C.

Archimedes (287–212 BC) was the greatest mathematician of antiquity and many consider him one of the greatest of all times. He was born in Syracuse on Sicily, educated in Alexandria and spent the rest of his life in Syracuse. He was very much part of a new culture initiated by Alexander the Great to forge an empire with Alexandria as a cultural and intellectual center.

Ptolemy and his dynasty that ruled Egypt after Alexander's demise created a cosmopolitan culture where people of different background and especially scholars were encouraged to settle in Alexandria. This mixing did not apply to rule and administration which was kept strictly in Greek and especially Macedonian hands. Knowledge and perspectives from different civilizations came together. Trade, navigation, production and engineering skills were vital, no longer something despised as practical matters done by a slave class so that free and educated men could spend their time on pure thought. Mathematics moved away from philosophy and reestablished ties with engineering.

Archimedes was a much respected man in his own lifetime, not for his mathematics but for his mechanical skill and inventions. King Hiero[n] II of Syracuse used his military devices such as catapults to defend Syracuse from Roman attacks. To us he may be best known for the story of how he realized when taking a bath that the rise in water level corresponded to the volume of the body immersed. He then allegedly shouted "Eureka! Eureka!" meaning "I have found it!…". Eager to share his discovery he ran out naked through the streets of Syracuse and then he used his discovery of measuring volume to determine whether a goldsmith had cheated with the gold in a crown for the king. Archimedes' principle "the buoyant force exerted on immersed bodies equals the weight of the fluid disposed by the bodies" is described in his book *On Floating Bodies.*

One of Archimedes' great achievements is his use of balancing torques to calculate areas, volumes and centers of gravity. Another method named Cavalieri's principle was an early precursor to integration, far ahead of its time. It was rediscovered by Chinese mathematicians in the 5th century and by European mathematicians in the 17th century. If two regions in a plane or space is bounded between two parallel lines/planes and if every parallel line/plane in between is intersected by line segments/cross sections of equal length/area then the areas/volumes of the regions are the same.

The methods are found in *The Method of Mechanical Theorem* (c. 250 BC), one of Archimedes' most important books. It was written in the form of a letter to Eratosthenes who was the chief librarian at the library of Alexandria. The content of the book was long believed to have been lost. Its survival is a fascinating story.

Archimedes works were compiled around 530 AD by Isidore of Miletus, the architect that was commissioned by East Roman emperor Justinian I to build Hagia Sopia. At that time it was to be Christian cathedral in Constantinople. Study of Archimedes flourished in the Byzantine Empire and copies of *The Method* were made around 950 AD. One manuscript ended up in Jerusalem some time after the destruction and looting of Constantinople in 1204 by western crusaders. The parchment was later reused for a Christian liturgical text by removal of old text. Centuries later in the 1840s, Constantin von Tischendorf, world leading German biblical scholar found the palimpsest in the monastery library at the orthodox church of the Holy Sepulcher in Jerusalem. He was intrigued by the mathematics visible beneath the text and brought back a page from it, presently kept in the Cambridge University Library. The book was lost but resurfaced in 1899 during an inventory. A transcription was made of some lines in the underlying text. The transcription reached the eyes of Johan Heiberg, Danish philologist and world authority on Archimedes. Heiberg made a study of the text in Constantinople in 1906 and confirmed that it was the work of Archimedes. Photographs were taken and an English translation was made. After the Greco-Turkish war 1919–1921 and expulsion of the Greek community the Archimedean palimpsest was lost again. It was later acquired by a Parisian who claimed to have bought it from a monk. He stored it secretly in a cellar where it was damaged by moist and molds. He died 1956 and in 1970 his daughter tried to sell it. Without luck she finally turned to Christie's in 1998 to sell it in a public auction. The ownership was contested and the manuscript was exposed to modern technology to reveal the underlying text.

New hidden texts were found, a commentary on the works of Aristotle from the 3rd century and a speech by an Athenian politician from the 4th century BC.

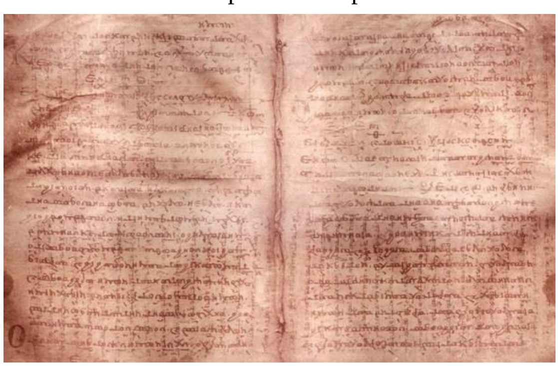
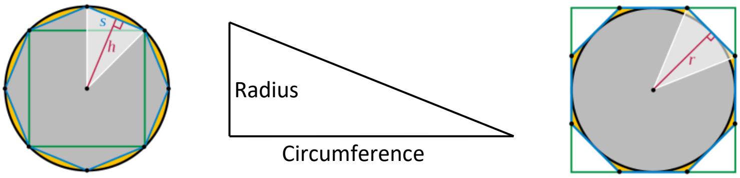


Fig. 2.9.15 Archimedes' palimpsest

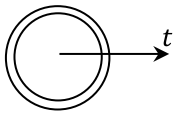
Eudoxus proved $A'/A = (r'/r)^2$ for circular areas and radii so $A = k \cdot r^2$. Archimedes then showed $k = \frac{\text{Circumference}}{\text{diameter}}$. The proof was by contradiction and used comparisons with inscribed and circumscribed regular polygons. Archimedes used them to put upper and lower bounds on k. This was mathematics of the Hellenistic era, it did not shun approximations and applications.



Calculating the area of a disc $C = \{(x, y) | x^2 + y^2 \leq r^2\}$ of radius r:

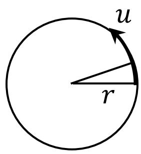
Integrating shells of area $2\pi t \cdot dt$:

$$\text{Area}(r) = \int_0^r 2\pi t dt = \pi r^2$$



Integrating triangles of area $(r \cdot du)/2$:

$$\text{Area}(r) = \int_0^{2\pi r} \frac{r}{2} du = \pi r^2$$

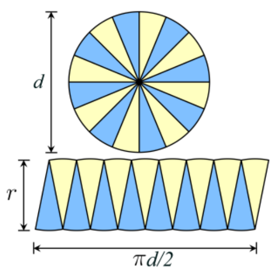


Integrating the unity function over C and changing the variables:

$$\text{Area(r)} = \iint_C 1 d(x, y) \begin{bmatrix} x = t\cos\theta \\ y = t\sin\theta \end{bmatrix} = \int_0^r \int_0^{2\pi} t d\theta dt = \begin{Bmatrix} \int_0^r 2\pi t dt \\ \int_0^{2\pi} \frac{r^2}{2} d\theta \end{Bmatrix} = \pi r^2$$

Defining π with the diameter in the denominator instead of the radius was a big historical mistake. The radius is the most natural parameter of a circle. With $\tau \equiv \pi/2$ we could replace 2π with τ in all formulas based on rotational symmetry. The angle for a full turn would be τ and the area would be the more natural:

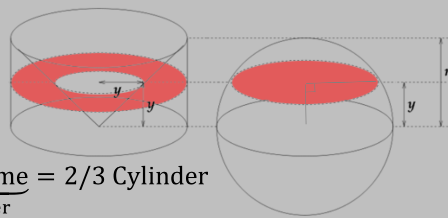
$$A(r) = \lim_{n\to\infty} n \cdot \frac{(r \cdot \tau/n) \cdot r}{2} = (\tau r^2)/2$$



A less intuitive rearrangement is Tarski's circle squaring. You can partition a disc into finitely many point sets and translate them to form a square of equal area. This was shown in 1989 with a proof that uses the axiom of choice so no actual decomposition is presented. The proof uses approximately 10^{50} non-measurable pieces. No meaningful area can be assigned to them.

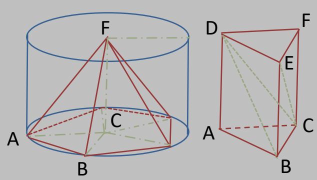
Volume of a sphere

Region 1: $A_1(y) = \pi r^2 - \pi y^2$
Region 2: $A_2(y) = \pi(r^2 - y^2)$



Adding cross-sections A_1:
Cylinder volume $-$ $\underbrace{\text{Cone volume}}_{1/3\ \text{Cylinder}}$ $= 2/3$ Cylinder

$A_1(y) = A_2(y)$ and Cavalieri's principle $\Longrightarrow$
Volume of sphere $= 2/3 \cdot$ Volume of circumscribed cylinder



Tetrahedron ABCD ≅ BCDE ≅ CDEF
since △ABD ≅ △BDE △BCE ≅ △CEF

The prism with bases ABC and DEF is the union of 3 disjoint congruent tetrahedra (ABCD, BCDE and CDEF) $\Longrightarrow$

Volume of tetrahedron ABCF $= \frac{1}{3} \cdot$ Volume of prism ABCDEF

$\Downarrow$

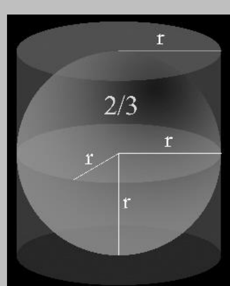
Volume of pyramid $= \frac{1}{3} \cdot$ Volume of prism with same base and height

$\Downarrow$ (taking limit of inscribed pyramids)

Volume of cone $= \frac{1}{3} \cdot$ Volume of circumscribed cylinder

Volume of sphere with radius r derived with integration:

$$2\int_0^r \pi(r^2 - y^2)dy = 2\pi\left[r^2 y - \frac{y^3}{3}\right]_0^r = \frac{4\pi r^3}{3}$$



π comes from the first letter of perimeter but Archimedes is not responsible for its unfortunate definition. The symbol and its definition became common practice when Euler adopted it in 1736. Among Archimedes achievements is a book *On Spirals* where he studies a spiral that has been named after him $r = a + b\theta$, in polar coordinates. The spiral can be used with ruler and compass to solve the problems of angle trisection and squaring the circle. In *The Quadrature of the parabola* he calculates the area of a parabolic segment by using a geometric series and in *Ostomachion* we find a puzzle of arranging pieces to form a square.

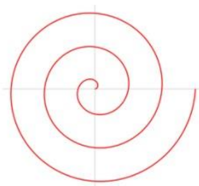
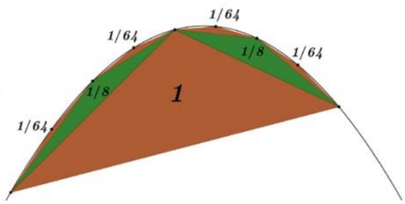



Fig 2.9.16 Archimedes' spiral, Parabolic segment and Ostomachion square puzzle

Another much harder puzzle is Archimedes' cattle problem that was found in a poem of a Greek manuscript. In the title it said that Archimedes sent the problem in a letter to be investigated by the mathematicians of Alexandria.

" Compute, O friend the number of cattle of the sun which once grazed upon the plains of Sicily, divided according to color into four herds, …"
They were white, yellow, black and dappled. There were bulls (W, Y, B, D) and cows (w, y, b, d), more bulls than cows. Their numbers were as:

$$\begin{aligned} W &= \left(\tfrac{1}{2}+\tfrac{1}{3}\right)B + Y \\ B &= \left(\tfrac{1}{4}+\tfrac{1}{5}\right)D + Y \\ D &= \left(\tfrac{1}{6}+\tfrac{1}{7}\right)W + Y \end{aligned} \qquad \begin{aligned} w &= \left(\tfrac{1}{3}+\tfrac{1}{4}\right)(B+b) \\ b &= \left(\tfrac{1}{4}+\tfrac{1}{5}\right)(D+d) \\ d &= \left(\tfrac{1}{5}+\tfrac{1}{6}\right)(Y+y) \\ y &= \left(\tfrac{1}{6}+\tfrac{1}{7}\right)(W+w) \end{aligned}$$

"If thou canst give, O friend, the number of each kind of bulls and cows, thou art no novice in numbers, yet can not be regarded as of high skill. Consider however, the following additional relations between the bulls of the sun."

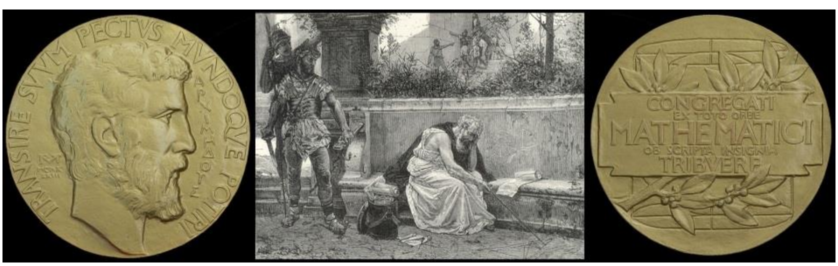
$W + B$ a square number
$D + Y$ a triangular number

"If thou hast computed these also, O friend, and found the total number of cattle, then exult as a conqueror, for thou hast proved thyself most skilled in numbers."

The smallest solution is so large that the mathematicians of Alexandria could hardly have solved the cattle problem nor could Archimedes, but he did write a book about large numbers. The cattle problem was solved in 1880 with a herd of $7.76 \ldots \cdot 10^{206\,554}$ animals. Archimedes wrote *The Sand Reckoner* to show that the number of grains of sand in the universe is not uncountable. He starts from a myriad that equals $10\,000$ and reaches $10^{8 \cdot 10^6}$. He even thinks of infinitely large numbers x that obey $x > n$ for every $n \in \mathbb{Z}$ and numbers $1/x$ that are infinitesimal but bigger than zero (Archimedean property p. 49).

Archimedes life had a dramatic end. The Roman republic was on the rise. On Sicily lived Romans, Greeks and Carthaginians with Phoenician roots. Their center was Carthage on the African coast. They were the masters of the sea. Rome and Carthage were in conflict. Rome defeated Carthage and took over Sicily in the first Punic war. Syracuse remained independent and in peace with Rome. Hieron II, king of Syracuse died in the beginning of the second Punic war and his successor Hieronymus thought that Rome would lose and allied himself with Carthage. Rome retaliated and captured the city after a two year long siege in 212 BC. Archimedes had built weapons to defend the city but Marcellus the Roman general considered him a valuable asset and ordered that he not be harmed. According to legend he was contemplating a mathematical drawing in the sand when he was disturbed by Roman soldiers before uttering his last words: “Do not disturb my circles!”

The Greek culture was revered by the Romans who never accomplished much in mathematics. Archimedes considered his proof of the volume of the sphere to be his greatest achievement and he wanted an inscription of it on his grave. Cicero, the Roman philosopher, statesman, orator and writer had heard stories of Archimedes’ tomb. When he served in Sicily he went to look for it. He found it, overgrown and in bad condition but with a sculpture showing a sphere inscribed in a cylinder.



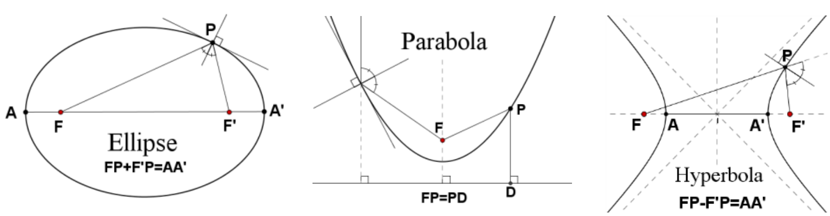
A testament to the greatness of Archimedes can be seen on the Fields medal, the most prestigious prize for a mathematician, awarded every four years. One side pictures Archimedes, the other side a sphere inscribed in a cylinder.

Struggle for control over Sicily had started the first Punic War against the Phoenicians (264–241 BC). It ended with Roman control over Sicily which became the first Roman province. The second Punic War (218–204 BC) which caused the death of Archimedes is famous for the march made by general Hannibal through Spain and southern France and across the Alps with a troop of 50 000 men 40 elephants. Very few survived the Alps and only one survived the war. A decisive Roman victory was accomplished by general Scipio in the battle of Zama near Carthage. The city was destroyed in the third Punic War (149–146 BC).

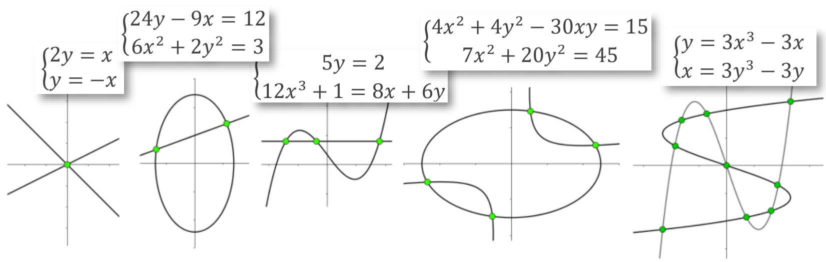
The Roman ascendance, their lukewarm interest in mathematics and their conquests to the east and south did not stop the progress of mathematics and science. Regions that had been influenced by Hellenism retained their culture and Alexandria continued to be an important intellectual center as part of a Roman province.

Aristarchus of Samos (c.310–c.230 BC), a contemporary of Archimedes was the first to present a heliocentric model of the universe with all known planetary orbits around the sun in the correct order. His measurement of the distance to the sun was based on a right triangle at half moon. With a second angle of 87º he put the Sun to close by a factor 400. This could have been proposed as a lower bound, Archimedes claims in *The Sand Reckoner* that Aristarchus used 89.5º. Another person to measure the distance to the sun was **Eratosthenes** (276–194 BC), close friend of Archimedes and chief librarian at Alexandria. He calculated the circumference of the earth based on the sun shining down from zenith in a well in Aswan, far south in Egypt while at the same time casting a shadow from a stick in Alexandria. He also calculated the tilt of the Earth's axis.

Another person in Alexandria that was of great importance for astronomy was **Apollonius** of Perga (c.260–c.190 BC). He provided the mathematical foundation for planetary orbits, both the flawed model of epicycles that was picked up by Ptolemy and the correct model of ellipses that was picked up by Kepler. Apollonius introduced the terms ellipse, parabola and hyperbola for different conic sections. These had been studied before by Euclid and others but it was Apollonius and his work *Conics* that almost completed the study.



Apollonius proved that two conics can intersect in at most four points. This theorem about two curves of 2nd degree is fertile ground for generalization.

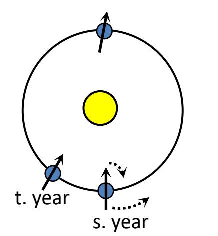


$$\bar{P}: \mathbb{R}^n \to \mathbb{R}^n \quad P_k(\bar{x}) = \sum_{i=1}^{N} \left(a_{ik} \prod_{j=1}^{n} {x_j}^{m_{ijk}} \right) \quad \begin{matrix} a_{ik} \in \mathbb{R} \\ m_{ijk} \in \mathbb{N}_0 \\ k \in \{1,2,\dots,n\} \end{matrix} \quad \bar{P}(\bar{x}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$Number\ of\ solutions \le \prod_{k=1}^{n} \max_{i \in \{1,\dots,N\}} \left(\sum_{j} m_{ijk} \right) \quad ?$$

His use of tangents, normals, curvature and reference lines inspired Descartes to introduce coordinate systems and analytic geometry. Leibnitz wrote “He who understands Archimedes and Apollonius will admire less the achievements of the foremost men of later times”.

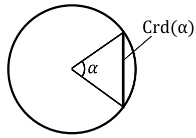
Another area with historical links to astronomy is trigonometry and especially spherical trigonometry. **Hipparchus** of Nicaea (c.190–c.120 BC) has been called the founder of trigonometry for his use of a trigonometric table. With centuries of astronomical data from Mesopotamia and his own observations he found a difference between a sidereal year, $365\frac{1}{4} + 1/444$ days and a tropical year, $365\frac{1}{4} - 1/300$ days. The sidereal year is based on the earth’s position in its orbit while the tropical year is based on the relative direction of the sun and the tilt of the earth axis. Seasons and days of solstice and equinox are linked to the tropical year. The difference is due to a slow rotation of the earth axis around the ecliptic axis directed tangentially to the plane of the solar system.



Precession per tropical year:
$\alpha = 360° \cdot (s - t)/t = 36''$
1° per century

Hipparchus conclusion was a precession of $36''$ per year, the correct value is $50.3''$ or $25\,800$ year for a full revolution. The mismatch corresponds to a few minutes on the length of a year. A calendar year is based on a tropical year to keep seasons from wandering. The historic name for this was “precession of the equinoxes”.

Hipparchus made a star catalog with 850 stars. For his astronomy he needed spherical trigonometry and for that he needed to calculate trigonometric functions like $\sin(\alpha)$. He adopted the Babylonian practice of 360 degrees for a full turn and made a trigonometric table based on the chord function $\mathrm{Crd}(\alpha)$ with values calculated in steps of 7.5° and interpolation for values in between. Values were calculated with classical Euclidean geometry corresponding to trigonometric identities.



$\mathrm{Crd}(\alpha) = 2r\sin(\alpha/2)$
$r = 3438 \rightarrow 2\pi r \approx 360 \cdot 60$
$\mathrm{Crd}(60°) = r$
$\sin\alpha/2 = \sqrt{(1 - \cos\alpha)/2}$
$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

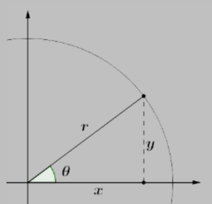
Hipparchus work in astronomy was continued by **Claudius Ptolemy** (100-170) three centuries later. He was Greek and a Roman citizen of Alexandria. To improve on Hipparchus work he expanded the chord table with calculated entries for every half degree. Ptolemy assembled his table by using a geometric theorem known as Ptolemy's theorem.

Ptolemy's works include three titles of historic importance. In order of increasing importance they were: *Tetrabiblos,* an astrological treatise, *Geography,* used by Columbus in his westerly search for India and *Almagest* which had a huge and long-lasting influence on the Byzantine, Islamic and European world. Its hegemony ended when Copernicus published *De revolutionibus orbium coelestium* in 1543.

Mathematical Treatise (Μαθηματικὴ Σύνταξις) was the original title of the Almagest. Later it became known as *Magna Syntaxis* (*The Great Treatise*). In the Arabic civilization it became known as *al-majisṭī* (المجسطي) which means *the Greatest*. When Europe woke up after a long time of ignorance after the fall of the Roman Empire they adopted the Latinized form of the Arabic title.Almagest contains a complete and detailed geocentric model of the cosmos. To explain the apparent motions of the heavenly bodies as seen from earth is a very messy affair. Ptolemy based his model on epicircles.

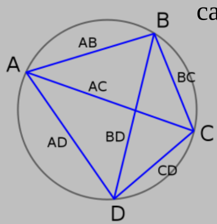
The reason he did not favor a heliocentric model was his belief in Aristotelean physics with earth firmly in the center and objects movements explained by their nature to fall down and not to move without external cause. The model worked well with Christian doctrine and Ptolemy's model based on a sound empirical approach became church dogma, the official truth for 1400 years.

Trigonometric identities and Ptolemy's theorem

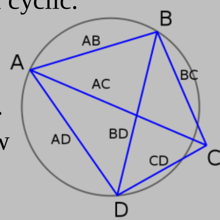


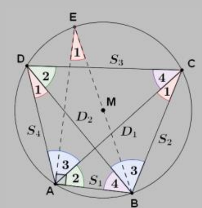
Pythagoras' theorem $x^2 + y^2 = r^2$ can be seen as a representation of $\cos^2\theta + \sin^2\theta = 1$.There are many other trigonometric identities useful for making a chord table hidden in a geometrical theorem proved by and named after Ptolemy.

Every triangle has a unique circumscribed circle. A quadrilateral that can be inscribed in a circle is called cyclic.

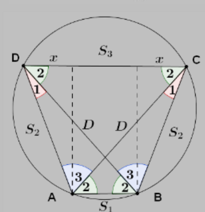


Ptolemy's theorem says:
AC·BD=AB·CD+BC·AD
for cyclic quadrilaterals ABCD.
Non-cyclic quadrilaterals follow Ptolemy's inequality:
AC·BD<AB·CD+BC·AD

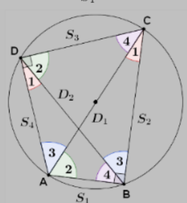




Let cyclic quadrilateral ABCD be inscribed in a circle of unit diameter.
$\angle BAE = 90° \Rightarrow S_1 = \sin\theta_1$ so $S_n = \sin\theta_n$
Ptolemy's theorem: $S_1 \cdot S_3 + S_2 \cdot S_4 = D_1 \cdot D_2$
(*) $\sin\theta_1 \cdot \sin\theta_3 + \sin\theta_2 \cdot \sin\theta_4 = \sin(\theta_1 + \theta_2) \cdot \sin(\theta_2 + \theta_3)$



Corollary 1: Let $\theta_2 = \theta_4$
$S_1 = S_3 - 2x = S_3 - 2S_2\cos(\theta_1 + \theta_2) \implies$
$S_3^2 - 2S_2S_3\cos(\theta_1 + \theta_2) + S_2^2 = D^2 \implies$
$D^2 = S_2^2 + S_3^2 - 2S_2S_3\cos(\theta_1 + \theta_2) \implies$
$c^2 = a^2 + b^2 - 2ab\cos\gamma$ for any triangle



Corollary 2: Let $\theta_1 + \theta_2 = 90°$, $\theta_3 + \theta_4 = 90°$
$x + y = 90° \Rightarrow \sin x = \cos y$
(*) → $\cos\theta_2 \cdot \sin\theta_3 + \sin\theta_2 \cdot \cos\theta_3 = \sin(\theta_2 + \theta_3)$
$\sin(\theta_2 + \theta_3) = \cos\theta_2 \cdot \sin\theta_3 + \sin\theta_2 \cdot \cos\theta_3$

Corollary 3: Let $\theta_3 = 90°$, $\theta_1 + \theta_2 + \theta_4 = 90°$
(*) → $\cos(\theta_2 + \theta_4) + \sin\theta_2 \cdot \sin\theta_4 = \cos\theta_4 \cdot \cos(-\theta_2)$
$\cos(\theta_2 + \theta_4) = \cos\theta_2 \cdot \cos\theta_4 - \sin\theta_2 \cdot \sin\theta_4$

All trigonometrical identities needed by Ptolemy for his chord table are contained in Ptolemy's theorem.

Up to this point there has not been much Greek algebra, Babylonians were more advanced 2000 years earlier. Things changed with **Diophantus**, also known as the father of algebra. He worked in Alexandria in the 3rd century and wrote a series of books on how to solve algebraic equations. They were titled *Arithmetica* which meant theory of numbers. The Greek name for the art of calculation or arithmetic was “logistica”. Practical calculations were made with an abacus, which is the word for “sand”. Writing in sand was probably a common way of writing and calculating. At some time pebbles may have been used since “calculus” means “pebble”.

Diophantus separated arithmetic and algebra from geometry, he started to use products with more than three factors and he treated fractions and integer as part of the same category, with integers a mere subset of all fractions. Negative and irrational numbers were however still unreal to Diophantus. A consequence was that he did not see 1 as a solution to $x^2 + 5 - 6x = 0$ since it corresponded to the negative root of $(x-3)^2 = 4$. In his algebraic mindset only the bigger of two roots to a quadratic equation was valid. He used methods for 2nd and 3rd degree equations similar to the Babylonians but went further and considered equations with several unknowns and he was the first to introduce a system of symbolic notation for algebra.

Alexandrian system of numbers	Diophantus’ symbols	Examples
α β γ δ ϵ ς ζ η θ 1 2 3 4 5 6 7 8 9 ι κ λ μ ν ξ o π ϙ 10 20 30 40 50 60 70 80 90 ρ σ τ υ ϕ χ ψ ω T 100 200 300 400 500 600 700 800 900	Reciprocal: $''$ Subtract: ⋔ Equals: ι^{σ} Unity: $\overset{o}{M}$ Variable: ς Inverse: χ Powers, 2nd to 6th: $\Delta^{\Upsilon}, K^{\Upsilon}, \Delta^{\Upsilon}\Delta,$ $\Delta K^{\Upsilon}, K^{\Upsilon}K$	21 : $\kappa\alpha$: 1305 : $,\overline{\alpha\tau\varepsilon}$ 1/3: γ'' $x^6 - 3x + 12x^2 - 4$: $K^{\Upsilon}K\overline{\alpha}\Delta^{\Upsilon}\overline{\iota\beta}$⋔$\varsigma\overline{\gamma}\overset{o}{M}\overline{\delta}$ $342 - 3/x = 9x^{-2}$: $\overset{o}{M}\overline{\tau\mu\beta}$⋔$\varsigma^{\chi}\overline{\gamma}\iota^{\sigma}\Delta^{\Upsilon\chi}\overline{\theta}$

The alphabet was repeated but preceded by a comma for numbers bigger than 999. A bar was used to avoid confusion with words. The symbols for equal, square and cube came from the two initial letters of the corresponding Greek words. M comes from the initial letter of monad, a Greek word for unity. One shortcoming of his system was that he only used one symbol for unknowns.

Diophantus studied polynomial equations with rational coefficients. If there were no rational solutions he considered the equation unsolvable. Polynomial equations where only integer solutions are of interest are called Diophantine. Some examples are seen on page 68. The most famous Diophantine equation is $a^n + b^n = c^n$. We met it on page 17 for $n = 2$, the solutions are called Pythagorean triplets. Any solution with a negative integer leads trivially to a solution with only positive integers by moving some term to the other side. By dividing with c^n we get $x^n + y^n = 1$ with $(x, y) \in (\mathbb{Q}^+, \mathbb{Q}^+)$. We are looking for rational points on a certain type of curve. Pierre de Fermat read Diophantus' *Arithmetica* and concluded that certain of the equations in the book had no solution. In the margin of his 1621 edition of *Arithmetica* he wrote: "If an integer n is greater than 2, then $a^n + b^n = c^n$ has no solutions in non-zero integers a, b, and c. I have a truly marvelous proof of this proposition which this margin is too narrow to contain." He never mentioned any proof in any of his later writings and the conjecture has become known as Fermat's Last Theorem. In 1995 it was finally turned into a true theorem by Andrew Wiles with the help of his former student Richard Taylor.

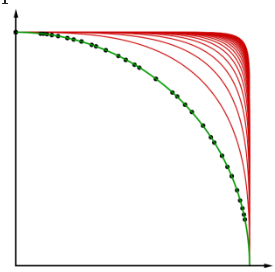


Fig. 2.9.16 Curves $x^n + y^n = 1$ with $n = 2$ solutions $\left(\frac{p^2-q^2}{p^2+q^2}, \frac{2pq}{p^2+q^2}\right)$.

Diophantus was one of the last of the great and innovative mathematicians of the Greek and Alexandrian eras, a period that was approaching its end. The final blow came with Arabic and Islamic conquest of the Roman province of Aegyptus in 641 but the decline had started long before.

Greek culture thrived during the Ptolemaic dynasty of Egypt but the Roman republic was on the rise. Octavian, later to be named Augustus and the first Roman emperor defeated his rival Marcus Antonius who was the lover of Queen Cleopatra, last ruler of the Ptolemaic dynasty. The Roman province of Egypt was established in 30 BC. The Romans respected Greek culture, they ruled an empire and played an important part of history for 1000 years but they did not produce one mathematician worth mentioning. Their skills were in other areas. They preferred concrete and practical applications, engineering projects were more their style.

Adoption of Christianity by rulers and people was another cause for the lowered prestige of mathematics. Links between astrology and astronomy and some branches of mathematics made things worse. Astrology was condemned by Roman emperors and Christians. Astrologers were called *mathematicii.* A mathematician by modern use was called a geometer and geometry came to dominate European mathematics into the Middle Ages and beyond. In the 19th century mathematicians began to call themselves mathematicians rather than geometers.

The library of Alexandria is a symbol of Greek culture and its destruction reflects the demise of Antiquity. Estimates of the number of scrolls in the library range from 40,000 to 400,000. All the important works by Greek mathematicians of Antiquity that have been lost were probably once placed in the library. Books were lost in stages. Caesar's troops started a fire during a siege of Alexandria that reached the library. To compensate for the loss Marcus Antonius gave Cleopatra a large number of scrolls as a wedding gift. The scrolls were taken from the 2nd biggest library in Pergamum, Turkey. Another period of damage occurred 270–275 AD when Emperor Aurelian suppressed a revolt against the Roman Empire led by Zenobia, a Syrian Queen that had conquered Egypt. A small part of the library located in the Serapeum may have survived. Serapeum was a temple devoted to the Hellenistic-Egyptian god Serapis. Paganism was declared illegal by Emperor Theodosius in 391 AD and the temples of Alexandria were closed by Pope Theophilius of Alexandria.

In these times of dusk for Greek civilization lived **Pappus** (c.290–c.350). He was far ahead of his contemporaries in a period of mathematical stagnation. Pappus' most famous theorem belongs to projective geometry, it states that for collinear points A,B,C and a,b,c there will be intersections x,y,z that are collinear.

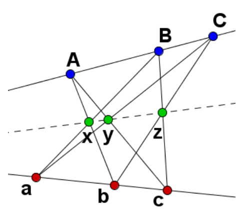


Fig. 2.9.17 Pappus' theorem

Pappus also wrote commentaries to Euclid's *Elements* and Ptolemy's *Almagest* and a work of eight books called *Mathematical Collections* that was a handbook that covered most of mathematics known at the time.

The Alexandrian period ends with comments and studies of earlier work. Theon's commented edition of Euclid's *Elementa* became the origin of later editions. His daughter **Hypatia** (c.360–415) was both mathematician, philosopher and astronomer. She was head of and taught at the Neoplatonic school of Alexandria. Hypatia's own work have been lost but we know she edited Ptolemy's *Almagest* and that she studied Apollonius' work on Conics.

Hypatia made astronomical instruments like the astrolabe and had access to astronomical data from centuries of observation. She must have been aware of Aristarchus' arguments for a heliocentric model and she may have noticed accumulating discrepancies between observations and predictions based on Ptolemy's geocentric model. It is not a far-fetched idea that she could have put two and two together and tried a heliocentric model with elliptic orbits.

Hypatia lived in a time of conflict between different groups in Alexandria. There was a Jewish group, a Christian group and there were those who hang on the old gods of Egyptian-Greek origin. Christianity appealed both to the poor and underprivileged with little interest for philosophy and astronomy and to people in power. Christianity became the official religion during the reign 379–395 of Emperor Theodosius I, paganism was outlawed, Hellenistic temples were destroyed and the Olympic games were banned.

It was the job of Roman governor Orestes to keep order. He was a moderate Christian who took advice from Hypatia. The Christians were led by Cyril, patriarch of Alexandria, not so moderate. Emotions were running high after episodes of mutual atrocities, conflict between Orestes and Cyril escalated and Hypatia became the target of anger and ill-will. A Christian mob of fanatics kidnapped her and took her to a church where they stripped, killed and tore her to pieces. Classical Antiquity was coming to an end, philosophy and learning was replaced with theology and holy scriptures.

Neoplatonism was the last philosophy of antiquity. It was founded by Plotinus (205–270) after extensive studies of many philosophical traditions. Ideas of a human soul and an eternal world of goodness and beauty governed by a unifying principle were central. Many religious thinkers were inspired and incorporated parts of it into their theologies. The Academy of Plato had not been active since 83 BC, it was revived in 410 AD as a centre of Neoplatonism and lasted until 529 AD when Justinian I, the Byzantine emperor closed everything that taught Hellenistic philosophy. Free thinking was over and the dark ages of Europe had begun.

Greek culture gave mathematics deductive proofs based on axioms and a view of mathematics as a basis for describing reality, something to be studied for its own sake. They excelled in geometry and introduced spherical and plane trigonometry. A downside of their insistence on secure foundations and reluctance towards infinity was their avoidance of irrational numbers in arithmetic and algebra. Irrational numbers were confined to geometry as part of continuous magnitudes and even then they preferred constructions starting from line segments and circles, but on the whole a great legacy.