A cyclic quadrilateral is a polygon where all four vertices lie on a circle. Cyclic polygons are exceptional except for triangles. Every triangle has a unique **circumscribed circle**. Its center called the **circumcenter** is located at the intersection of the three perpendicular bisectors. The proof of existence is similar to the proof for a unique **inscribed circle**, centered at the intersection of the three angular bisectors, the **incenter**.

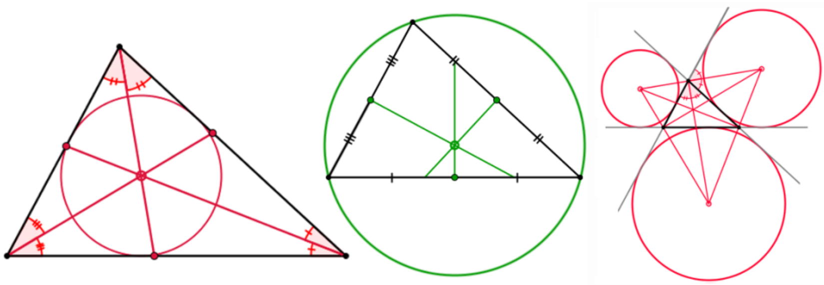
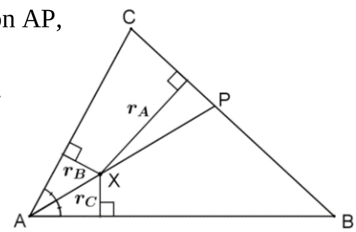


Fig. 3.6.1 Incircle, circumscribed circle and excircles with angle/segment bisectors.

Proof. (Inscribed circle)



Any inscribed circle must be centered on AP, the angle bisector of ∠BAC.
Let point X vary along AP from A to P.
$r_B(\mathrm{X}) = r_C(\mathrm{X}) \equiv r_{B,C}(\mathrm{X})$
$r_{B,C}(X)$ is strictly increasing.
$r_A(\mathrm{X})$ is strictly decreasing.

$$\left.\begin{aligned} 0 = r_{B,C}(\mathrm{A}) < r_A(\mathrm{A}) \\ r_{B,C}(\mathrm{P}) > r_A(\mathrm{P}) = 0 \end{aligned}\right\} \text{Continuity} \rightarrow \exists!\,\mathrm{X}\ \mathrm{s.t.}\ r_{B,C}(\mathrm{X}) = r_A(\mathrm{X})$$ ■

Alternative proof:
Intersection of bisectors to ∠A and ∠B gives a point X s.t.
$r_B(\mathrm{X}) = r_C(\mathrm{X})$ and $r_C(\mathrm{X}) = r_A(\mathrm{X}) \rightarrow r_A = r_B = r_C$ at point X, which is the unique center of the inscribed circle of △ABC.

Two other examples where three lines in a triangle intersect in a single point are the **orthocenter** (altitudes) and the **centroid** (apex–midpoint lines).

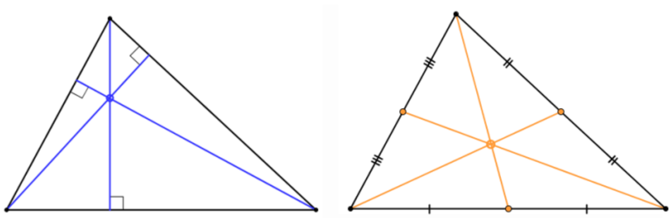
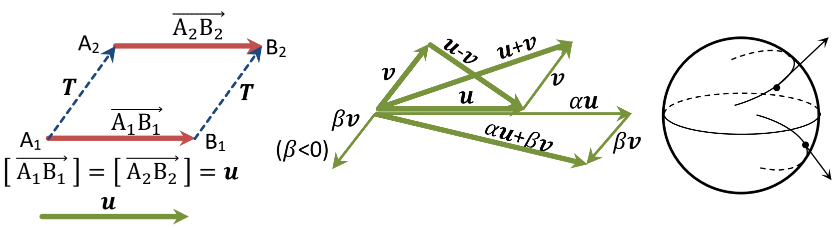


Fig. 3.6.2 Orthocenter and centroid of a triangle.

The centroid is the geometric center of an object. If an object is divided into subparts that approach zero size then the arithmetic mean of their positions will approach the geometric center. When the mean is weighted by local density you get the mass center from which the object balances in any orientation under the influence of a uniform gravitational field. The centroid of a two-dimensional shape can be found by intersecting plumb-lines.

Vector algebra is a useful approach to show that the three lines will intersect in a single point. A vector can be visualized as a displacement, a directed segment from one point A to another point B, $\overrightarrow{\mathrm{AB}}$. The word "vector" comes from Latin and means carrier. In epidemiology a vector carries and transmits pathogens from one organism to another.

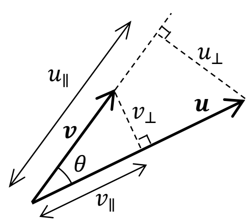
$\overrightarrow{\mathrm{AB}} + \overrightarrow{BC}$ equals $\overrightarrow{AC}$ but what about $\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{CD}}$? In a flat Euclidean space we solve it by parallel displacement. Directed segments related by a translation represent one and the same vector. Translate them to a common origin and form a parallelogram or put one vector after the other. Vectors have direction and length, $\|\boldsymbol{v}\| \geq 0$ but no particular origin. They can be multiplied by scalars, $c \in \mathbb{R}$. $c < 0$ means the opposite direction and $\|c \cdot \boldsymbol{v}\| = |c| \cdot \|\boldsymbol{v}\|$. Comparing vectors with different origin in a curved space requires extra care.



An algebra over a field is a vector space equipped with a bilinear product. Multiplication of vectors is called **scalar product** or **dot product**.

Definition. (Scalar product)

$\boldsymbol{u} \cdot \boldsymbol{v} \equiv \pm\|\boldsymbol{u}\| \cdot v_{\parallel}$



Positive sign if $\boldsymbol{u}$ and $\boldsymbol{v}_{\parallel}$ point in the same direction and negative sign if they point in opposite directions.
$\boldsymbol{v}_{\parallel}$ is the projection of $\boldsymbol{v}$ along the axis of $\boldsymbol{u}$ and
$\boldsymbol{v}_{\perp}$ is the projection along the perpendicular axis.

$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \cdot \|\boldsymbol{v}\| \cdot \cos\theta = \pm\|\boldsymbol{v}\| \cdot u_{\parallel} = \boldsymbol{v} \cdot \boldsymbol{u}$
$\boldsymbol{u} \cdot (\alpha\boldsymbol{v} + \beta\boldsymbol{w}) = \alpha\boldsymbol{u} \cdot \boldsymbol{v} + \beta\boldsymbol{u} \cdot \boldsymbol{w}$

$\boldsymbol{u} \perp \boldsymbol{v} \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = 0$ ($\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular iff their dot product is zero)

The geometric vectors satisfy the properties of a **vector space** $(V, +, \cdot)$ over a field $\mathbb{F}$. Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in V and let a and b be scalars in $\mathbb{F}$, then:

$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$	Associativity of addition
$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$	Commutativity of addition
$\exists \boldsymbol{0} \in V$ s.t. $\forall \boldsymbol{v} \in V: \boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$	Existence of zero vector
$\forall \boldsymbol{v} \in V\ \exists \boldsymbol{w} \in V\ s.t. \boldsymbol{v} + \boldsymbol{w} = \boldsymbol{0}$	Existence of additive vector inverse
$a(b\boldsymbol{v}) = (a \cdot_{\mathbb{F}} b)\boldsymbol{v}$	Scalar and field product compatible
$1 \cdot \boldsymbol{v} = \boldsymbol{v}$	Multiplicative identity in $\mathbb{F}$, also for V
$a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$	Distributivity of $\cdot_V$ with respect to $+$
$(a +_{\mathbb{F}} b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$	Distributivity of $\cdot$ with respect to $+_{\mathbb{F}}$

A space $(V, +, \cdot)$ with an **inner product** $\langle \cdot, \cdot \rangle: V \times V \to \mathbb{R}$ (or $\mathbb{C}$) like the dot product, is called an **inner product space** if the inner product satisfies:

$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$	Conjugate symmetry
$\langle a\boldsymbol{u} + b\boldsymbol{v}, \boldsymbol{w} \rangle = a\langle \boldsymbol{u}, \boldsymbol{w} \rangle + \beta\langle \boldsymbol{v}, \boldsymbol{w} \rangle$	Linearity of $\langle \cdot, \cdot \rangle$ in first argument
$\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geq 0 \land (\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0})$	Positive-definiteness

A vector space $(V, +, \cdot)$ over a field $\mathbb{F} \subseteq \mathbb{C}$ with a **norm** $\|\cdot\|: V \to \mathbb{R}$ like the length of a vector, is called a **normed vector space** if the norm satisfies:

$\|a\boldsymbol{v}\| = \lvert a \rvert \cdot \|\boldsymbol{v}\|$	Absolute homogeneity / scalability
$\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$	Triangle inequality
$\|\boldsymbol{v}\| = 0 \Rightarrow \boldsymbol{v} = \boldsymbol{0}$	Only the zero vector has norm zero

Proof. (Centroid)

Line through A and D, $\overrightarrow{\mathrm{AD}} = \overrightarrow{\mathrm{AB}} + \frac{1}{2}\overrightarrow{\mathrm{BC}}$:

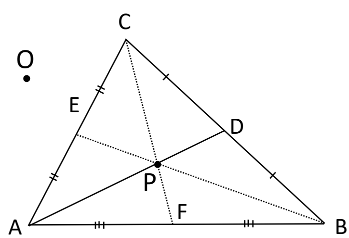
$\overrightarrow{\mathrm{OA}} + k \cdot \left(\overrightarrow{\mathrm{AB}} + \frac{1}{2}\overrightarrow{\mathrm{BC}}\right) =$

$\overrightarrow{\mathrm{OA}} + k \cdot \left(\overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}} + \frac{1}{2}\left(\overrightarrow{\mathrm{OC}} - \overrightarrow{\mathrm{OB}}\right)\right) =$

$(1-k)\overrightarrow{\mathrm{OA}} + \frac{k}{2} \cdot \left(\overrightarrow{\mathrm{OB}} + \overrightarrow{\mathrm{OC}}\right)$

The point P where $1 - k = \frac{k}{2} \Leftrightarrow k = \frac{2}{3}$ is symmetric with respect to A, B and C.

$\mathrm{P} = \frac{1}{3}\left(\overrightarrow{\mathrm{OA}} + \overrightarrow{\mathrm{OB}} + \overrightarrow{\mathrm{OC}}\right)$



The symmetry means that AD, BE and CF must intersect in this point. ■

∴ The center of mass and geometric center of △ABC lies on the intersection of the apex-midpoint lines at a distance $2/3$ from the apex.

A proof that also altitudes intersect in a single point is left as an exercise.

The circumcenter, centroid and orthocenter are collinear. They all lie on the Euler line mentioned on page 132. Some other special triangle points also lie on the line but the incenter is on the Euler line only for isosceles triangles.

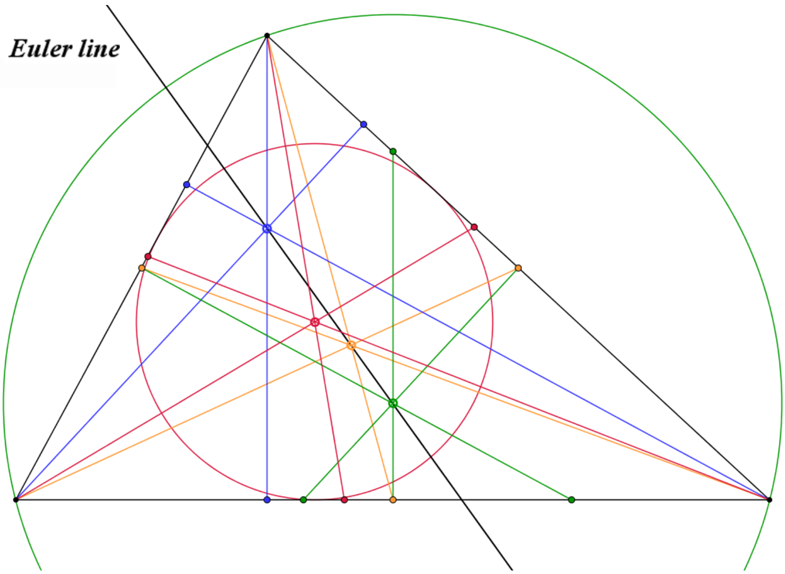


Fig. 3.6.3 Euler line with orthocenter, centroid and circumcenter. Incenter is off line.

Synthetic geometry, the one without coordinates and formulas did not end with Euclidean geometry. Over time there came other axiomatic formulations and other types of geometries to explore:

- Affine geometry, Euclidean geometry without distances and angles.
- Projective geometry, studies properties unchanged by projections,
  incorporates points at infinity that make space close in on itself.
- Absolute geometry, Euclidean geometry without any parallel postulate.
- Euclidean geometry, one line through a point P parallel to line L.
  Hilbert's 20 axioms, modern treatment of Euclidean geometry (1899).
  Tarski's formulation, from 1959 based on fist-order logic.
- Non-Euclidean geometry
  Spherical geometry, based on the surface of a sphere.
  Elliptic geometry, no lines through P parallel to L.
  Hyperbolic geometry, more than one line through P not intersecting L.

Coordinates were introduced in Apollonius' work on conics where he used perpendicular reference lines and measured distances to them. Modern use of coordinates dates back to Descartes (1596–1650) (Latinized: Cartesius), hence the names **Cartesian coordinate system** and Cartesian product of sets.

$A\times B = \{(a,b) | a \in A \land b \in B\}$, $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) | a_k \in \mathbb{R}, 1 \le k \le n\}$

Another introduction of Descartes is the use of the letters x, y and z for unknown variables and a, b and c for parameters or known variables and superscripts for exponents. For these and other achievements Descartes has become known as the father of analytical geometry. He bridged the gap between algebra and geometry.

Descartes introduced coordinate systems in an appendix titled *La Géometrie* in *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences*. The book also contains the famous quotes “Je pense, donc je suis” and “Cogito ergo sum”. The book was published in 1637 but it would take a long time before Euclidean style geometry lost its grip on mathematical teaching. Euclid’s *Elements* from 300 BC was still used a textbook in the 20th century; partly due to conservatism and partly due to its precise logic and austere beauty.

When Isaac Newton in 1687 published Principia, his work on mechanics and gravitation (in Latin: *Philosophiae Naturalis Prinicipa Mathematica*) he used Euclidean style geometry to get the most impact, his own methods with differential calculus where probably to unfamiliar for an audience trained in classical geometry. Euler’s work made the methods of Leibnitz and Newton with analytical geometry and differential calculus the standard of physics.

$\mathbb{R}^n$ has a very natural vector space structure with,
addition: $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) \equiv (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$
and scalar multiplication: $\alpha(a_1, a_2, \dots, a_n) \equiv (\alpha a_1, \alpha a_2, \dots, \alpha a_n)$
$\mathbb{R}^n$ is sometimes considered as a set and sometimes as a vector space.

There is also a natural set of vectors from which all others can be expressed:

$$\left.\begin{matrix} \boldsymbol{e}_1 = (1,0,0,\dots,0) \\ \boldsymbol{e_2} = (0,1,0,\dots,0) \\ \vdots \\ \boldsymbol{e_n} = (0,0,0,\dots,1) \end{matrix}\right\} \to (a_1, a_2, \dots, a_n) = a_1\boldsymbol{e_1} + a_2\boldsymbol{e_2} + \dots + a_n\boldsymbol{e_n}$$

These vectors form a **base**. The **dimension** of a space equals the number of base vectors. The coefficients of the basis vectors are called **coordinates**.

$\mathbb{R}^n$ also has a natural dot product, $\langle\cdot,\cdot\rangle: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, it is linear in both slots so $\langle \sum_{i=1}^n a_i\, \boldsymbol{e_i}, \sum_{j=1}^n b_j\, \boldsymbol{e_j} \rangle = \sum_{1\le i,j\le n} a_i b_j \langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle$. The natural choice is to set $\langle \boldsymbol{e_i}, \boldsymbol{e_j} \rangle = \delta_{i,j}$. With this choice and our original interpretation of dot product as $\boldsymbol{e_i} \cdot \boldsymbol{e_j} = \|\boldsymbol{e_i}\| \cdot \|\boldsymbol{e_j}\| \cdot \cos\theta$ we get basis vectors that are of unit length and perpendicular to each other.

The Euclidean space of two or three dimension on which we place our shapes and solids resembles $\mathbb{R}^2$ or $\mathbb{R}^3$ but there is a subtle difference. The Euclidean space consists of points rather than vectors and it has no special point like the zero vector of a vector space. I will not give the formal definition of a Euclidean space $\mathbb{E}^n$, it's a bit technical. From points you get displacements and equivalence classes of these corresponds to vectors.

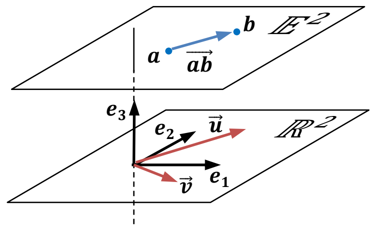


Fig. 3.6.4 Real Euclidean space, $\mathbb{E}^2$

The Cartesian coordinate system can be a great visual aid for gaining mathematical insights that can be used when doing formal proofs and getting a feel for functions, by looking at their graphs.

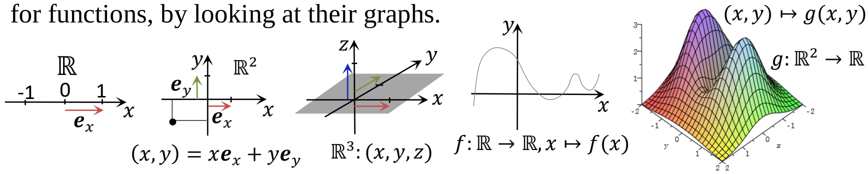


Fig. 3.6.5 Real Cartesian coordinate systems and graphs of functions

The coordinate system for one dimension is just the number line. From a mathematical viewpoint there is no favored direction but the real world is not symmetric. The natural choice is a horizontal line with increasing values in the same direction as we read and write. For each new dimension there is a new perpendicular direction and a new choice of positive direction. For two dimensions this direction is up. Angles are measured positive in a counter-clockwise direction and negative for clockwise rotations. In three dimensions the right hand is used as a model for positive orientation of axis, curl the fingers from the x-arrow towards the y-arrow and the thumb will point in the positive z-direction. For $\mathbb{R}^n, n \geq 4$ there are no physical objects to guide us and symmetry is restored.

It's not just physical objects that differentiate between these two possible orientations. The laws of nature for elementary particles are asymmetric with respect to a change of parity, where all three space axes are reversed (P). If a parity transformation is combined with a charge reversal (C) then the laws are less asymmetric but still not symmetric (CP violation). Only when charge, parity and time are all reversed, do we 'expect' the laws of nature to be the same. No incidence of CPT-violation has been observed. Classical mechanics is symmetric with respect to both parity and time reversal whereas a quantum gravity theory such as loop quantum gravity has CPT-violation.

A general rotation $\boldsymbol{x} \mapsto R_t(\boldsymbol{x})$ is characterized by a continuous displacement over time, $t \in [0,1]$ that starts from the identity $R_0(\boldsymbol{x}) = \boldsymbol{x}$ with a pivot point $R_t(\boldsymbol{P}) = \boldsymbol{P}$ and fixed relative distances $\|\boldsymbol{x} - \boldsymbol{y}\| = \|R_t(\boldsymbol{x}) - R_t(\boldsymbol{y})\|$. Euler showed that in three dimensions the final position is characterized by a rotation axis, also known as the Euler axis and a perpendicular rotation plane. This is not the case in higher dimensions.

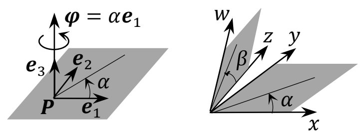


Fig. 3.6.6 Examples of rotations in three and four dimensions.

Note that three-dimensional rotation vectors $\boldsymbol{\varphi}$ do not form a vector space if $\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2$ means the rotation vector of rotation $\boldsymbol{\varphi}_1$ followed by rotation $\boldsymbol{\varphi}_2$. It does not commute, $\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 \neq \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1$:

$$90°\boldsymbol{e}_1 + 90°\boldsymbol{e}_2\colon\ R_{90°\boldsymbol{e}_2}\left(R_{90°\boldsymbol{e}_1}(\boldsymbol{e}_3)\right) = R_{90°\boldsymbol{e}_2}(-\boldsymbol{e}_2) = -\boldsymbol{e}_2$$
$$90°\boldsymbol{e}_2 + 90°\boldsymbol{e}_1\colon\ R_{90°\boldsymbol{e}_1}\left(R_{90°\boldsymbol{e}_2}(\boldsymbol{e}_3)\right) = R_{90°\boldsymbol{e}_1}(\boldsymbol{e}_1) = \boldsymbol{e}_1$$

By forming a rotation plane of two coordinate axes and rotating 180° we can change their directions. Repeated use of this in $\mathbb{R}^n$ leads to a reduction from 2^n to 2 rotationally invariant choices of axis directions, positive and negative orientation (or handedness). This leads to a classification of manifolds into orientable and non-orientable.

Fig. 3.6.7 Non-orientable surfaces, Möbius strips and Klein bottle.

Handedness of a coordinate system does not change under local movement but if transported on a global path, it may return with the opposite handedness. A surface or n-dimensional manifold with this property is non-orientable. The Möbius strip is non-orientable and one-sided whereas the torus is orientable and two-sided, inside and outside. A clock with hands going clockwise (CW) will return after one lap on the Möbius strip with hands going counterclockwise (CCW). The Möbius strip has a border that is a simple and closed curve, when cut along the center line it transforms into a band with two full twists instead of the half-twist in the Möbius strip, it has become orientable. Check and see! What happens if you cut off center?

The Möbius strip was discovered independently by two German mathematicians in 1858, August Möbius and Johann Listing.

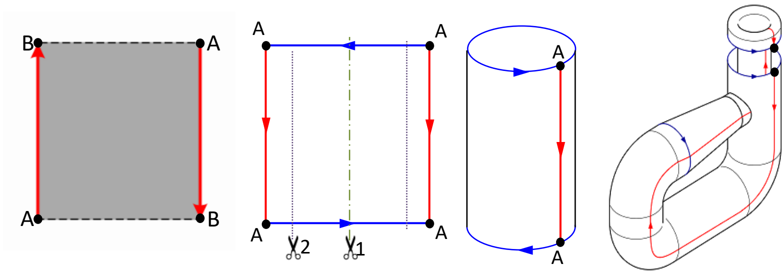


Fig. 3.6.8 Construction of Möbius strip and Klein Bottle.

The Möbius strip can be constructed from a rectangle by joining opposite edges after a half-twist so that the arrows match each other. The Klein bottle, another non-orientable surface is constructed in a similar way. The Klein bottle has no border, just like the surface of a sphere. A physical Klein bottle with no self-intersection can be made in four dimensions but not in three. What becomes of the bottle if you cut it up along cut no.1 and what happens if you cut along line no.2?

Non-orientable manifolds have some curious properties. Imagine the non-orientable 3-dimensional solid Klein bottle which is the space obtained from a solid cylinder by gluing the top to the bottom, not directly which gives a solid torus but via a reflection in a line.

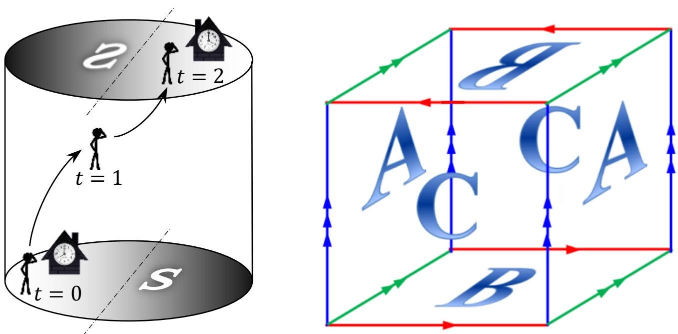


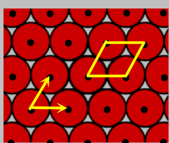
Fig 3.6.9 Non-orientable 3-manifolds, with and without boundary

A traveler in a non-orientable universe could return home after a long journey as a mirror image of himself. A right-handed person would come home as a left-handed person and if he or she had a watch with hands, they would by others be seen to run counterclockwise. The traveler would only see external changes, everybody and everything had turned into their mirror image.

Sphere packing, part 2

Part 1 of "Sphere packing" covered the 3-dimensional case of finding the densest packing of non-overlapping spheres. Generalization is often fruitful so how dense can you pack n-spheres, $S^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| = r\}$ in $\mathbb{R}^{n+1}$. The proportion of space filled by the spheres and their interior in a particular arrangement will depend on the volume considered but as the volume goes to infinity the density will approach a well-defined limit.

The 1-dimensional case (n=0) is trivial, with 0-spheres consisting of the endpoints of an interval the coverage becomes 100%. The plane case has an obvious candidate with centers arranged in a regular triangular pattern.

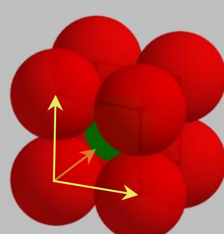


The density of this hexagonal packing is $\pi/2\sqrt{3} \approx 0.91$. It was proved optimal among regular arrangements by Lagrange in 1773. A rigorous proof that no irregular arrangement could do better was given in 1940 by László Tóth. The optimal 3D-pattern from part 1 has a density of $\pi/3\sqrt{2} \approx 0.74$.

A regular arrangement is described by a **lattice** $\Lambda = \{\sum_{i=1}^{n} a_i \boldsymbol{v_i} \,| a_i \in \mathbb{Z}\}$ where $\boldsymbol{v_i}$ form a basis that divides $\mathbb{R}^n$ into an array of parallelepipeds or fundamental domains. The lattice points have translational symmetries that corresponds to the group $\mathbb{Z}^n$.

The E_8 lattice in $\mathbb{R}^8$ and the Leech lattice in $\mathbb{R}^{24}$ are optimal packings with extraordinary symmetries. Other dimensions are less symmetrical; the densest known packings in some dimensions are not even regular. The maximal number of spheres that can touch a given sphere is only known for 1, 2, 4, 8 and 24 dimensions. This is the kissing number problem and the answers are 2, 6, 12, 24, 240 and 196560. Density decreases with n, the exceptionally effective packing in $\mathbb{R}^{24}$ has density $\pi^2/12! \approx 0.002$.

The cubic lattice in $\mathbb{R}^n$ with ON-base is a rather ineffective packing with a surprising property that makes it even worse than you might first think. Put a sphere in the center $(½, ½, \dots, ½)$ and maximize its radius so that it kisses the surrounding spheres. Explore how the radius varies with the dimension and you will find something quite unexpected.



With an orthonormal base $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$ of positive orientation we can express vectors with coordinates, calculate dot products, decide the length of a vector and determine angles between vectors:

$$\overrightarrow{\mathrm{OP}} = \sum_{i=1}^{n} x_i \boldsymbol{e}_i \equiv (x_1, x_2, \dots, x_n)$$

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i\, \boldsymbol{e}_i \cdot \sum_{j=1}^{n} y_j\, \boldsymbol{e}_j = \sum_{k=1}^{n} x_k y_k$$

$$\|\boldsymbol{x}\|^2 = \sum_{i=1}^{n} x_i^2 \qquad \cos\theta = \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\|\boldsymbol{x}\|\cdot\|\boldsymbol{y}\|}$$

Vector pairs in higher dimension span a space of two dimensions or less where angles are a familiar concept.

In three dimensions another bilinear operator is possible, not into the scalar numbers but into the vector space itself $\times\colon (\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}^3, \left(\vec{A}, \vec{B}\right) \to \vec{A} \times \vec{B}$. It's called the vector product. The length of $\vec{A} \times \vec{B}$ equals the area spanned by $\vec{A}$ and $\vec{B}$, the direction is perpendicular to the plane spanned by $\vec{A}$ and $\vec{B}$ in such a way that $\vec{A}$, $\vec{B}$ and $\vec{A} \times \vec{B}$ form a right-handed triple:

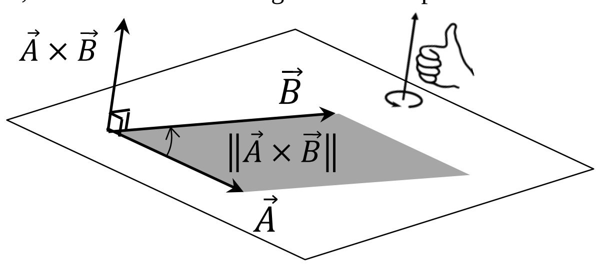
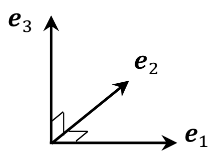


Fig. 3.6.9 Vector product

Two-dimensional spaces can also have a vector product if the space is embedded in a 3-dimensional space by adding a third dimension. For an orthonormal (ON) basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ of positive orientation:

$$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3 \quad \boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1 \quad \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2$$

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \sum_{k=1}^{n} \varepsilon_{ijk}\, \boldsymbol{e}_k$$

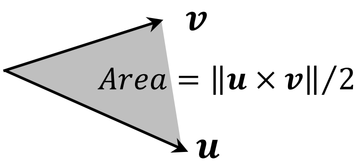


The Levi-Civita symbol $\varepsilon_{i_1 i_2 \dots i_n}$ with n indices, $i_k \in \{1,2,\dots,n\}$ is defined by $\varepsilon_{12\dots n} = 1$, two equal indices make it zero and by being totally antisymmetric under permutation of the indices:

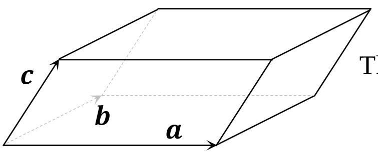
$$\varepsilon_{i_1 i_2 \dots i_n} = (-1)^p \varepsilon_{12\dots n}\ ,\ \ p = \text{number of interchanges to unscramble } i_1 \dots i_n.$$

$$\boldsymbol{x} \times \boldsymbol{y} = \sum_{i=1}^{3} x_i \boldsymbol{e}_i \times \sum_{j=1}^{3} y_j \boldsymbol{e}_j = \sum_{i,j,k=1}^{3} \varepsilon_{ijk} x_i y_j \boldsymbol{e}_k$$

$$(x_1, x_2, x_3) \times (y_1, y_2, y_3) = (x_2 y_3 - x_3 y_2\ , x_3 y_1 - x_1 y_3\ , x_1 y_2 - x_2 y_1)$$



Areas of triangular regions given in coordinate form are now easy to calculate.



The volume of a parallelepiped spanned by $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is:

$$\|\boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b})\| = \|\boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})\| = \|\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})\|$$

Vectors, dot products and vector products are essential in physics, with or without coordinates. Most basic is the positional vector $\boldsymbol{r}$, with coordinates (x, y, z) in some operationally defined coordinate system. A true vector is independent of the coordinate system. It is not just a collection of three numbers, the numbers must transform in well-defined ways when looking at the vector from different reference frames.

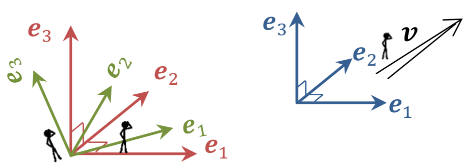


Fig. 3.6.10 Different reference frames, one rotated and one moving away.

For reference frames in relative motion this transformation will involve time. In special relativity position vectors are no longer true vectors; time must be included as a fourth component to make a space-time vector (x, y, z, t) or as a pre-component in a time-space vector $(t, x, y, z) \sim (x_0, x_1, x_2, x_3)$. These 4-dimensional vectors are called four-vectors or Lorentz vectors. Magnitudes should always be independent of the reference frame. For a 4-vector:

$$\|\boldsymbol{x}\|^2 = -x_0^2 + x_1^2 + x_2^2 + x_3^2 \ \text{ or } \ x_0^2 - x_1^2 - x_2^2 - x_3^2 \ \text{ depending on taste.}$$

3-vectors from classical physics where space and time are separated, $\mathbb{R}^3 \times \mathbb{R}$:

	Translation	Rotation	Vector fields
	Linear-	Angular-	defined everywhere
Displacement:	$\Delta \boldsymbol{r}$	$\Delta \boldsymbol{\varphi}$	Electric: $\boldsymbol{E}(\boldsymbol{r}, t)$
Velocity:	$\boldsymbol{v} = \frac{\Delta \boldsymbol{r}}{\Delta t}$	$\boldsymbol{\omega} = \frac{\Delta \boldsymbol{\varphi}}{\Delta t}$	Magnetic: $\boldsymbol{B}(\boldsymbol{r}, t)$
Acceleration:	$\boldsymbol{a} = \frac{\Delta^2 \boldsymbol{r}}{(\Delta t)^2} = \frac{\Delta \boldsymbol{v}}{\Delta t}$	$\boldsymbol{\alpha} = \frac{\Delta \boldsymbol{\omega}}{\Delta t}$	Force on charge q
Jerk:	$\boldsymbol{j} = \frac{\Delta^3 \boldsymbol{r}}{(\Delta t)^3} = \frac{\Delta \boldsymbol{a}}{\Delta t}$	$\boldsymbol{\zeta} = \frac{\Delta \boldsymbol{\alpha}}{\Delta t}$	with velocity $\boldsymbol{v}$
Momentum:	$\boldsymbol{p} = m\boldsymbol{v}$	$\boldsymbol{L} = I\boldsymbol{\omega}$	in E-M fields $\boldsymbol{E}$ and $\boldsymbol{B}$:
Force/Torque:	$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$	$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt}$	$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

These vectors are classified into one of two groups, polar/true vectors and axial/pseudo vectors depending on how they behave under reflection in a plane. The linear and $\boldsymbol{E}$ vectors are true vectors whereas the angular and $\boldsymbol{B}$ vectors are pseudo vectors. They all transform as vectors under rotation but the pseudo-vector coordinates get an extra sign change under reflection.

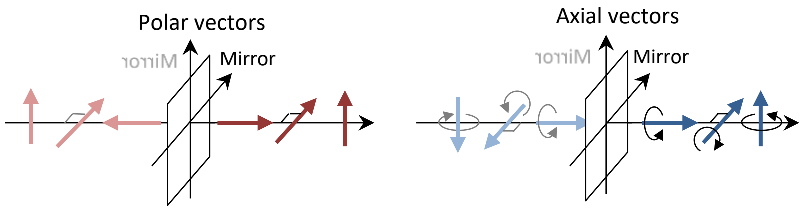
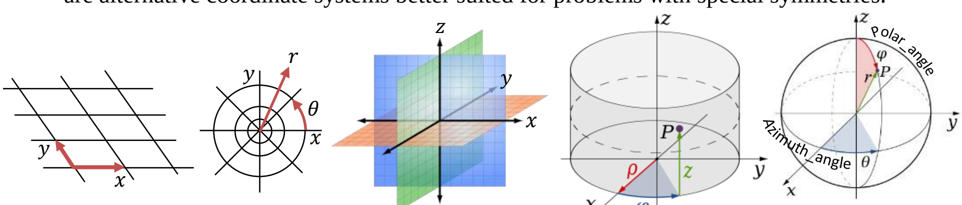


Fig. 3.6.11 Vectors behaving differently under reflection.

The Cartesian coordinates system is not the only way to express points in $\mathbb{R}^n$. There are alternative coordinate systems better suited for problems with special symmetries.



Skewed (x, y) coordinates, Non-orthogonal

Polar (r, θ) coordinates $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

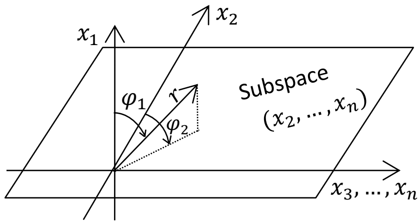
Cartesian (x, y, z) coordinates

Cylindrical (ρ, φ, z) coordinates $\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases}$

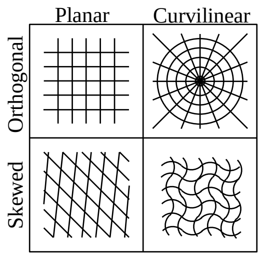
Spherical (r, θ, φ) coordinates $\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$

Spherical coordinates suffer from variation in conventions. In physics both the letters and the order of the polar and azimuth angles are interchanged so that the spherical harmonics $Y_l^m(\theta, \varphi)$ of physics have $\theta \leftrightarrow$ polar angle and $\varphi \leftrightarrow$ azimuth angle.

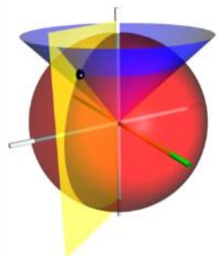
Spherical coordinates generalize to n dimensions $(r, \varphi_1, \varphi_2, \ldots, \varphi_{n-2}, \theta)$ where $\varphi_k \in [0, \pi]$ is the polar angle with the x_k-axis and $\theta \in [0,2\pi)$ is the azimuthal angle in the (x_{n-1}, x_n)-plane, so called hyperspherical coordinates.



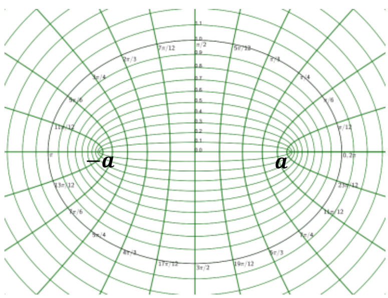
$$\begin{cases} x_1 = r\cos\varphi_1 \\ x_2 = r\sin\varphi_1\cos\varphi_2 \\ x_3 = r\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\ \quad\vdots \\ x_{n-2} = r\sin\varphi_1\cdots\sin\varphi_{n-3}\cos\varphi_{n-2} \\ x_{n-1} = r\sin\varphi_1\cdots\sin\varphi_{n-3}\sin\varphi_{n-2}\sin\theta \\ x_n = r\sin\varphi_1\cdots\sin\varphi_{n-3}\sin\varphi_{n-2}\cos\theta \end{cases}$$



Coordinates can be planar, curvilinear, orthogonal or skewed depending on the properties of the $(n-1)$-dimensional coordinate "surfaces" where one coordinate is held fixed.



Elliptic coordinates



$$\begin{cases} x = a\cosh\mu\cos\nu \\ y = a\sinh\mu\sin\nu \end{cases} \quad\leftrightarrow\quad x + iy = a\cosh(\mu + i\nu)$$

$$\mu \geq 0: \quad \frac{x^2}{a^2\cosh^2\mu} + \frac{y^2}{a^2\sinh^2\mu} = \cos^2\nu + \sin^2\nu = 1$$

Coordinate curves of fixed μ are ellipses.

$$0 \leq \nu < 2\pi: \quad \frac{x^2}{a^2\cos^2\nu} - \frac{y^2}{a^2\sin^2\nu} = \cosh^2\mu - \sinh^2\mu = 1$$

Coordinate curves of fixed ν are hyperbolae.

The coordinate systems shown so far have singled out a special point as the origin, often denoted O. The Euclidean space $\mathbb{E}^n$ has no such special point. Other such examples are the **projective spaces** consisting of the set of lines through the origin of a vector space. For $\mathbb{R}^n$ and $\mathbb{C}^n$ they are $\mathbb{RP}^n$ and $\mathbb{CP}^n$. $\mathbb{RP}^2$, the 2-dimensional real projective plane can be seen as:

- All lines passing through the origin in $\mathbb{R}^3$.
- The surface of a sphere with antipodal points identified.
- Equivalence classes: $(\mathbb{R}^3 \setminus \{\mathbf{0}\})/\sim$ with $\boldsymbol{x} \sim \boldsymbol{y}$ iff $\boldsymbol{x} = \lambda \boldsymbol{y}$, $\lambda \in \mathbb{R} \setminus \{0\}$.

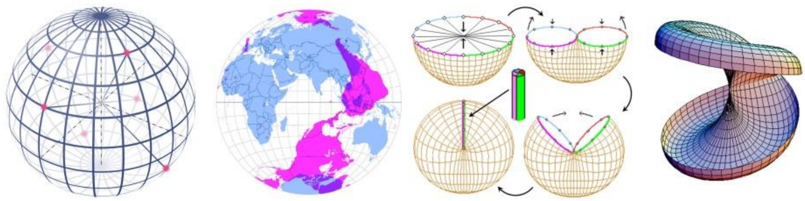


Fig. 3.6.12 Antipodal points on a globe and the surface of $\mathbb{RP}^2$.

Projective spaces can be equipped with **homogeneous coordinates**, one more coordinate than the dimension. The equivalence class of (x, y, z) in $\mathbb{RP}^2$ is denoted $[x{:}\, y{:}\, z] = [x/\lambda{:}\, y/\lambda{:}\, z/\lambda]$. Such coordinates were introduced 1827 by August Möbius the inventor of the Möbius strip. $\mathbb{RP}^2$ can be construed as a Möbius strip where the single edge has been glued to a disk. It takes four dimensions to show the surface without self-intersection. The real projective plane is an example of a compact (finite area, no boundary), non-orientable, 2-dimensional surface. $\mathbb{RP}^{2k}$ is non-orientable whereas $\mathbb{RP}^{2k+1}$ is orientable, $\mathbb{RP}^1$ is the equivalent to a circle. Projective spaces are the natural arena for **projective geometry**.

Renaissance art is full of projective geometry. A projection from a 3D-scene to the painter's eye brings the scenery into a 2D-image. Parallel lines on the ground that goes inwards meet at a horizontal line. Projective geometry studies those properties that are unaffected by projective transformations. Distances and angles change but collinearity and cross-ratio for collinear points $(\|\overline{\mathrm{AC}}\| \cdot \|\overline{\mathrm{BD}}\|)/(\|\overline{\mathrm{AD}}\| \cdot \|\overline{\mathrm{BC}}\|)$ stay the same.

Fig 3.6.13 Perspective projection in Renaissance art

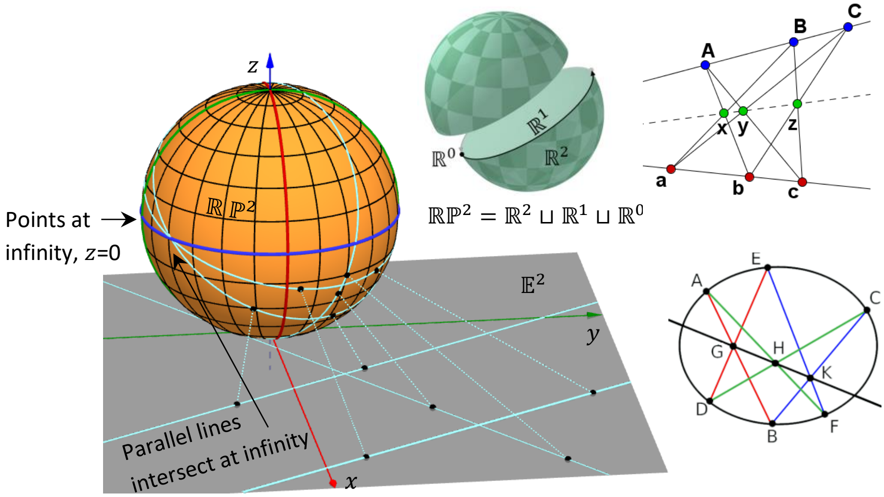


Fig. 3.6.14 Projective plane, straight lines, great circles and projective geometry.

Projecting from the center of sphere S^n through the lower hemisphere onto the Euclidean space $\mathbb{E}^n$ on which the sphere stands shows a link between $\mathbb{E}^n$ and the projective space $\mathbb{RP}^n$. The projective space is a compactification of the Euclidean space, accomplished by adding points at infinity.

$$\mathbb{RP}^n = \mathbb{R}^n \sqcup \underbrace{\mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^0}_{\text{Points at } \infty \,=\, \mathbb{RP}^{n-1}} \quad (\sqcup \text{ is the disjoint union})$$

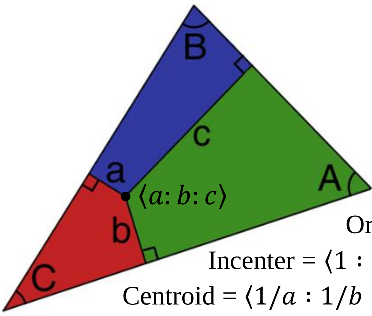
Straight lines on the plane correspond to great circles on the sphere. Every straight line has a unique point at infinity where its great circle crosses the equator and parallel lines share a common point at infinity.

With the unit sphere centered at $\mathbf{0}$ and the Euclidean plane at $z = -1$ we get homogeneous coordinates $[x: y: -1]$ for the points in $\mathbb{E}^2$. A line through the center $[0: 0: -1]$, $ax + by = 0 \Leftrightarrow (x, y) \cdot (a, b) = 0 \Leftrightarrow (x, y) \perp (a, b)$ has points parametrized as $(x, y) = t \cdot (b, -a)$. In homogeneous coordinates this becomes $[bt: -at: -1] = [b: -a: -1/t]$ which goes to $[b: -a: 0]$ as t goes to infinity. Points at infinity are those $[\alpha: \beta: \gamma]$ with $\gamma = 0$ and $(\alpha, \beta) \neq (0,0)$.

Formulas and theorems are often more symmetric and simple when expressed in projective geometry. In $\mathbb{RP}^2$ every pair of unique lines have a unique intersection; no exception is needed for parallel lines. Figure 3.38 shows two theorems that belong to projective geometry. The first is Pappus's theorem on collinearity from page 73. The second is Pascal's theorem, a generalization of Pappus's theorem. It is also known as the *Hexagrammum Mysticum Theorem*.

The *Hexagrammum Mysticum Theorem* states that if six arbitrary points in arbitrary order are chosen on a conic (ellipse, parabola or hyperbola) and joined into a hexagon then the three points where opposite sides meet (extended if necessary) are collinear. A degenerate conic with two lines gives Pappus's theorem.

Other examples of homogeneous coordinates are barycentric coordinates and **trilinear coordinates**. The latter are based on a given triangle, △ABC:



$\langle a\!:b\!:c\rangle$ is the point P, with distances a, b and c from sides BC, CA and AB respectively. Only their ratios matter, $\langle \lambda a\!:\lambda b\!:\lambda c\rangle \wedge \lambda > 0$ is the same point.

Orthocenter = $\langle 1/\cos \mathrm{A} : 1/\cos \mathrm{B} : 1/\cos \mathrm{C}\rangle$

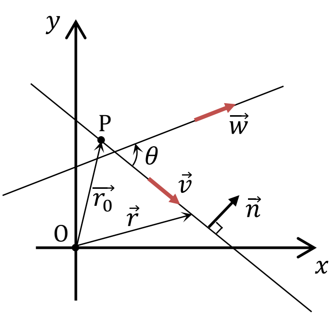
Incenter = $\langle 1 : 1 : 1\rangle$ A=$\langle 1\!:0\!:0\rangle$ A-excenter = $\langle -1\!:1\!:1\rangle$

Centroid = $\langle 1/a : 1/b : 1/c\rangle$ B=$\langle 0\!:1\!:0\rangle$ B-excenter = $\langle 1\!:-1\!:1\rangle$

Circumcenter = $\langle \cos \mathrm{A} : \cos \mathrm{B} : \cos \mathrm{C}\rangle$ C=$\langle 0\!:0\!:1\rangle$ C-excenter = $\langle 1\!:1\!:-1\rangle$

With a coordinate system in place the scene is set for **analytic geometry**, also known as coordinate geometry. Analytic geometry can turn a geometric problem into an algebraic problem. It is a tool of huge importance for physics and engineering but it can also be a temptation and a menace that turns an elegant and simple proof from synthetic geometry into a monstrosity with complicated formulas that hide all conceptual clarity.

In the rest of this section Cartesian coordinates $(x, y, z \ldots)$ or $(x_1, x_2, \ldots)$ will be assumed. Curves and surfaces can be specified with equations and in-equalities. For k equations $f_i(x_1, x_2, \ldots, x_n) = 0$, $i \in I = \{1, \ldots, k\}$ there is a **solution set** $\{\boldsymbol{x} \in \mathbb{R}^n | f_i(\boldsymbol{x}) = 0 \ \wedge i \in I\}$, built from intersections and often an object of $n - k$ dimensions (the number of parameters needed to describe the object). Each new equation reduces the dimension by one unit.



Coordinate equation for a line in the plane:

$\vec{r} = \overrightarrow{\mathrm{OP}} + t \cdot \vec{v}\,,\ t \in \mathbb{R}$

$\vec{w} \leftrightarrow e^{i\theta}(v_x + iv_y)$

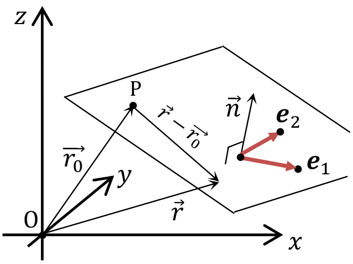
$\vec{n} \leftrightarrow \underbrace{e^{i\pi/2}}_{i}(v_x + iv_y) \leftrightarrow (-v_y, v_x)$

$\vec{n} = (a, b)\,, \vec{r_0} = (x_0, y_0)\,, \vec{r} = (x, y)$

$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \Leftrightarrow$

$ax + by = ax_0 + by_0$

$ax + by = c$ or $y = kx + m$ if $b \neq 0$



Coordinate equation for a plane in space:

$\vec{r} = \overrightarrow{\mathrm{OP}} + t_1\boldsymbol{e}_1 + t_2\boldsymbol{e}_2\,,\ t_1, t_2 \in \mathbb{R}$

$\vec{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2 = (a, b, c)$

$\vec{r_0} = (x_0, y_0, z_0)\,, \vec{r} = (x, y, z)$

$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \Leftrightarrow$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\mathrm{ax + by + cz = d}$

The intersection of three planes is found from a **system of linear equations**.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_2 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases} \leftrightarrow \left.\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}\right| \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \leftrightarrow \left.\begin{matrix} a'_{11} & a'_{12} & a'_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a'_{33} \end{matrix}\right| \begin{matrix} b'_1 \\ b'_2 \\ b'_3 \end{matrix}$$

The primed symbols can be found by **Gaussian elimination**, named after C.F. Gauss from the 19th century. The procedure was used in China already in the 2nd century. The method reduces the original equations to equations with the same set of solutions. The three operations below are used repeatedly to get zeros in the lower left triangle which makes the system easy to solve. Solve the equations from the bottom up and introduce free parameters (s, t) for variables when possible. The solution set can be empty, a unique point or it can be a one- or two-dimensional Euclidean subspace, given by a particular solution $(\boldsymbol{u})$ and vectors $(\boldsymbol{v}, \boldsymbol{w})$ that span the space.

- Swap positions of two rows.
- Multiply a row by a scalar≠0.
- Add a scalar multiple of one row to another row.

$\rightarrow$

$\boldsymbol{x} \in \emptyset$ or

$\boldsymbol{x} \in \{\boldsymbol{u}\}$ or

$\boldsymbol{x} \in \{\boldsymbol{u} + s \cdot \boldsymbol{v} | s \in \mathbb{R}\}$ or

$\boldsymbol{x} \in \{\boldsymbol{u} + s \cdot \boldsymbol{v} + t \cdot \boldsymbol{w} | s, t \in \mathbb{R}\}$

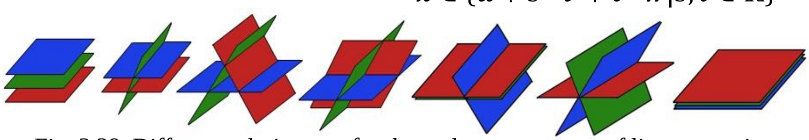
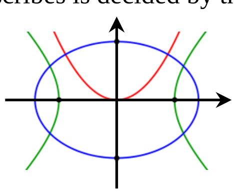
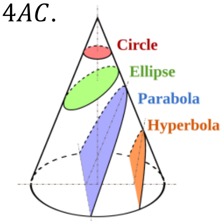


Fig. 3.39 Different solution sets for three planes or system of linear equations.

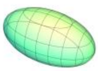
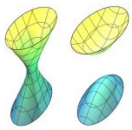
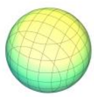
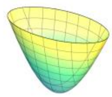
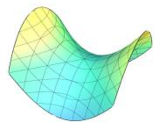
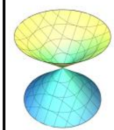
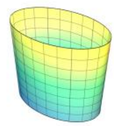
Lines, planes and hyperplanes are described by polynomials of degree one: $\sum_{i=1}^{n} P_i x_i + R = 0$. The next objects to study are those described by polynomials of degree two: $\sum_{i,j=1}^{n} Q_{ij} x_i x_j + \sum_{i=1}^{n} P_i x_i + R = 0$. In $\mathbb{R}^2$ these are, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. They are all curves formed by the intersection of a plane and a double-cone. What conic section the equation describes is decided by the discriminant $d = B^2 - 4AC$.



- $d = 0$: Parabola
- $d < 0$: Ellipse
- $d > 0$: Hyperbola



The algebraic equation of degree two in three dimensions leads after some suitable change of variables to a group of different basic forms with center at the origin and coordinate axis along the perpendicular principal axes. Below are some examples of such quadric surfaces of varying generality.

Ellipsoid	Hyperboloid	Sphere	Paraboloid		Circular cone	Elliptic cylinder
			Elliptic	Hyperbolic		
						
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\pm 1$	$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{a^2}=1$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$	$\frac{x^2}{a^2}+\frac{y^2}{a^2}-\frac{z^2}{b^2}=0$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

3.7 Calculus

In this section of basic mathematics we will look into calculus, basic analysis. Later chapters will be devoted to specific branches of mathematics, the type that you could meet in a typical university course such as algebra number theory, and complex analysis. We will then have the opportunity to return to our original problem of decimal expansion.

3.7.1 Limits

This is not our first encounter with limits; on page 95 we used it to enlarge the domain of numbers from rational numbers to real numbers. A strict definition of limits is essential for analysis. Without rigor things can easy go astray as they often did in the early days of analysis.

The first to propose limits instead of infinitesimals as the basis for analysis was d'Alembert (1717–1883). General recognition of the limit concept came with its appearance in the very influential textbook "Cours d'Analyse" from 1821 by Augustin-Louis Cauchy. The concept was still only formulated in words. The formal definition with ϵ and δ that many students of mathematics has struggled with derives from Bernard Bolzano (1781–1848). The limit definition given here comes from Karl Weierstrass (1815–1897). He also introduced the notation $\lim_{x \to x_0}$ which Hardy modified by putting the arrow below lim in another classic "A Course of Pure Mathematics" from 1908.

Let us assume a function f with range in $\mathbb{R}$ and domain in $\mathbb{R}$ that contains a punctured neighborhood of x_0 ($0 < |x - x_0| < d$).

Definition.

f has the limit y_0 in x_0 if for every $\epsilon > 0$ there is a $\delta > 0$ such that:

$$0 < |x - x_0| < \delta \Longrightarrow |f(x) - y_0| < \epsilon$$

This is written as $\lim\limits_{x \to x_0} f(x) = y_0$

Note that the function can have any value in x_0 or not even be defined in x_0. This definition has a natural generalization to any function $f: X \to Y$ with range and domain in metric spaces, just replace $|p - q|$ with $d(p, q)$, the distance between p and q in the given metric space. For normed vector spaces such as $\mathbb{R}^n$ with $||\vec{x}||^2 = \sum_{i=1}^{n} x_i^2$ we have $d(\vec{p}, \vec{q}) = ||\vec{q} - \vec{p}||$. If a limit exists it is unique since different points have disjoint neighborhoods. Based on the distance function and its properties (p. 95) we can form a series of useful notions for sets in metric spaces:

Interval: A connected set on $\mathbb{R}$, with endpoints $[a, b] \equiv \{x | a \leq x \leq b\}$ or without endpoints $]a, b[\equiv (a, b) \equiv \{x | a < x < b\}$, a and b can be $\pm\infty$. Semi-open intervals like $[a, b[$ or $(-\infty, b]$ have only one endpoint. Note that open ends has two alternative notations, not to be used together.

Ball: Part of a metric space that has a center p and a radius r. A ball can be open $B_r(p) = \{x | d(x, p) < r)\}$ or closed $B_r[p] = \{x | d(x, p) \leq r)\}$.

Sphere: The surface or boundary of a ball. The n-sphere of radius r may be defined as an embedding in $\mathbb{R}^{n+1}$, $S^n = \{x \in \mathbb{R}^{n+1}: ||x|| = r\}$.

Interior: Interior point p of a set $\mathcal{S}$ in a metric space is a point with an open ball $B_r(p) \subseteq \mathcal{S}$. The interior of $\mathcal{S}$=int$(\mathcal{S}) = \mathcal{S}^0$ is its interior points.

Open set: A set $\mathcal{S}$ in which each point is an interior point, $\mathcal{S} = \mathcal{S}^0$.

Closed set: A set $\mathcal{S}$ whose complement is an open set.

Neighborhood of a point p in a metric space $\mathcal{M}$ is a set $\mathcal{V}$ that includes an open set $\mathcal{U}$ containing p. It's a deleted or punctured neighborhood if p is excluded from the neighborhood.

Limit point of a set $\mathcal{S}$ is a point p such that every neighborhood of p contains at least one point of $\mathcal{S}$ other than p itself.

Closure of a set $\mathcal{S}$ is the set plus its limit points=cl$(\mathcal{S}) = \bar{\mathcal{S}}$.

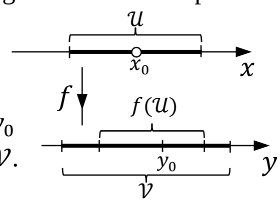
Boundary of a set $\mathcal{S}$ is the set of points called boundary points. These are inside the closure of $\mathcal{S}$ but outside the interior of $\mathcal{S}$. It is denoted by $\partial\mathcal{S}$.

Dense set: A subset $\mathcal{S}$ of $\mathcal{M}$ is dense if every point of $\mathcal{M}$ belongs to $\mathcal{S}$ or is a limitpoint of $\mathcal{S}$. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$.

Isolated point of a subset $\mathcal{S}$ is a point in $\mathcal{S}$ with a neighborhood that does not contain any other point of $\mathcal{S}$.

Discrete set: A set consisting only of isolated points.

There is an equivalent definition of limit in terms of neighborhoods and open sets that don't use the language of ϵ and δ:

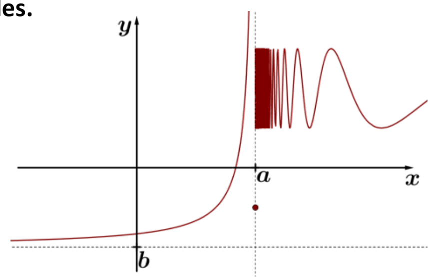


Definition.
f has the limit y_0 in x_0 if for every neighborhood $\mathcal{V}$ of y_0 there is a punctured neighborhood $\mathcal{U}$ of x_0 s.t. $f(\mathcal{U}) \subseteq \mathcal{V}$.

This definition of limits can be generalized beyond the domain of metric spaces where distance functions generate open sets. A more general approach is to start from a collection of open sets that satisfies certain properties that capture our idea of how open sets should behave. In this more general setting no distances are needed and might not even be possible to define and still we could speak of limits and continuity. This is the foundation of topology and it will be dealt with in another chapter.

With minor alterations in the (ϵ, δ)-definition of $f: \mathbb{R} \to \mathbb{R}$ there can be limits when approaching from the "left" $x \to a^-$, from the "right" $x \to a^+$, going towards $-\infty$ or $+\infty$ or when the limit goes to $-\infty$ or $+\infty$. Sequences based on functions $f: \mathbb{Z}^+ \to \mathbb{R}$ have limits defined in a similar manner.

Examples.



$$\lim_{x \to a^-} f(x) = \infty \;\leftrightarrow\; \forall M \exists \delta > 0 \;:\; a - \delta < x < a \Rightarrow f(x) > M$$

$$\lim_{x \to -\infty} f(x) = b \;\leftrightarrow\; \forall \epsilon > 0 \exists m \;:\; x < m \Rightarrow |f(x) - b| < \epsilon$$

$$\lim_{n \to \infty} a_n = b \;\leftrightarrow\; \forall \epsilon > 0 \exists N \;:\; n > N \Rightarrow |a_n - b| < \epsilon$$

Infinity symbols are usually a sign of an underlying limiting process.

$$\sum_{k=1}^{\infty} a_k = S \;\leftrightarrow\; \lim_{n \to \infty} \underbrace{\sum_{k=1}^{n} a_k}_{s_n} = S \;\leftrightarrow\; \forall \epsilon > 0 \exists N : n > N \implies |s_n - S| < \epsilon$$

Irrational numbers like $\pi = 3.14159\ldots$ can be seen as limits in this way:

$$A = a_0.a_1a_2\ldots = a_0 + \sum_{n=1}^{\infty} a_n \cdot 10^{-n}$$

Proving that a limit exists usually means finding an m, N or δ for a given ϵ that make the final equality true, i.e. a function $m(\epsilon), N(\epsilon)$ or $\delta(\epsilon)$.

Example.

Show that $\lim_{x\to \underbrace{4}_{x_0}} \underbrace{\frac{x^4-1}{x-1}}_{f(x)} = \underbrace{4}_{y_0}$

$f(x) = \dfrac{x^4-1}{x-1} = (x^2+1)(x+1)$ if $x \neq 1$

$|f(x) - y_0| = |x^3+x^2+x-3| = |x-1| \cdot |x^2+2x+3|$

Choose a $K \in \mathbb{R}^+, 0 < |x-1| < K \Longrightarrow \exists M \in \mathbb{R}^+ :\ |x^2+2x+3| < M \Longrightarrow$

$0 < |x-1| < \underbrace{\min\left(K, \frac{\epsilon}{M}\right)}_{\delta} \Longrightarrow |f(x) - y_0| < \epsilon$ $\boxed{\delta(\epsilon) = \min\left(K, \frac{\epsilon}{M}\right)}$ ■

Limits of real-valued functions obey some useful theorems.

Theorem. (Algebraic limit theorem)

If $\lim_{x\to c} f(x) = A$ and $\lim_{x\to c} g(x) = B$ then

$\lim_{x\to c}\big(f(x)+g(x)\big) = A+B$ $\lim_{x\to c}\big(f(x)-g(x)\big) = A-B$

$\lim_{x\to c}\big(f(x)\cdot g(x)\big) = A\cdot B$ $\lim_{x\to c}\big(f(x)/g(x)\big) = A/B$ if $B \neq 0$

Proof. (of the first statement, the others are left as exercises)

For a given $\epsilon' > 0$ there is $\delta_1 > 0$ and $\delta_2 > 0$ s.t.

$$\left.\begin{array}{l} 0 < |x-c| < \delta_1 \Rightarrow |f(x)-A| < \epsilon' \\ 0 < |x-c| < \delta_2 \Rightarrow |g(x)-B| < \epsilon' \end{array}\right\} \Longrightarrow \text{(by triangle inequality)}$$

$0 < |x-c| < \min(\delta_1,\delta_2) \Rightarrow |(f(x)-A) + (g(x)-B)| < 2\epsilon'$

For any $\epsilon > 0$, let $\epsilon' = \frac{\epsilon}{2}$ then we can use $\delta(\epsilon) = \min(\delta_1,\delta_2)$. ■

The rules work for one-sided limits, for $c = \pm\infty$ and for infinite limits with some extensions of arithmetic such as $A + \infty = \infty$ if $A \neq -\infty$, $A \cdot \infty = \infty$ if $A > 0$ and $A/\infty = 0$ if $A \neq \pm\infty$.

Theorem. (Limit of composite function)

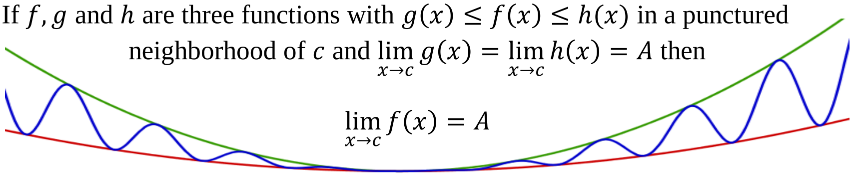
$$\lim_{x\to c} g(x) = A \quad \wedge \quad \lim_{x\to A} f(x) = B \quad \Longrightarrow \quad \lim_{x\to c} f\big(g(x)\big) = B$$

For this implication to be true at least one of two extra conditions must hold. $f(A)$=B (f continuous at $x = A$) or $g(x) \neq A$ in a deleted neighborhood of c.

Theorem. (Squeeze theorem)

If f, g and h are three functions with $g(x) \leq f(x) \leq h(x)$ in a punctured neighborhood of c and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = A$ then

$$\lim_{x \to c} f(x) = A$$



A function $f: \mathbb{R} \to \mathbb{R}$ is continuous if its graph can be drawn from $-\infty$ to $+\infty$ without lifting the pen, which means that $\lim_{x \to c} f(x) = f(c)$ for any $c \in \mathbb{R}$. With the convention that a function is continuous in isolated points of its domain the general definition for a continuous function $f: X \to Y$ between two metric spaces is.

Definition.

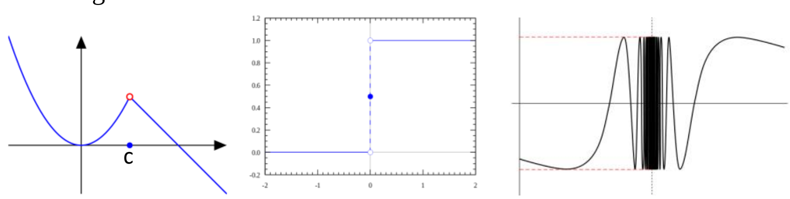
f is continuous if $\lim_{x \to c} f(x) = f(c)$ for every limit point c in D_f.

The set of continuous mappings from X to Y is written $C(X, Y)$ and for continuous functions $f: \mathbb{R} \to \mathbb{R}$ the usual notation is $f \in C^0(\mathbb{R})$. $C^k(\mathbb{R})$ is for functions differentiable k times and with $f^k \in C^0(\mathbb{R})$. Continuous functions preserves limits of sequeces, $\lim_{n \to \infty} x_n = c \Rightarrow \lim_{n \to \infty} f(x_n) = f(c)$.

With these rules and starting from the continuity of $f(x) = C$ and $g(x) = x$ it follows that the polynomials $P(x) = \sum_{n=0}^{N} a_n x^n$ and the rational functions $h(x) = P(x)/Q(x)$ are continuous functions ($D_h = \{x \in \mathbb{R} | Q(x) \neq 0\}$).

In terms of neighborhood and sets a function $f: X \to Y$ is continuous if the preimage $f^{-1}(V) \coloneqq \{x \in X | f(x) \in V\}$ of every open set V is an open set. This definition is at the core of topology where distance is secondary and where continuity will depend on the choice of open sets.

A function $f: \mathbb{R} \to \mathbb{R}$ can fail to be continuous at a point c in different ways. Discontinuities can be classified as removable, jump-type or essential. If $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(c)$ it can be removed by redefining $f(c)$, Not to be confused with removable singularities where c in not part of D_f. In a jump discontinuity the one-sided limits exist, they are finite but unequal. Remaining cases are essential discontinuities.

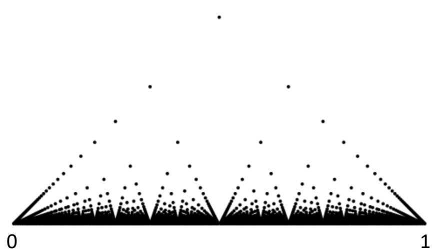


The points of discontinuity need not be isolated points.

Dirichlet's function:

$$D(x) = \begin{cases} 1 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{Q} \end{cases}$$

D is nowhere continuous.

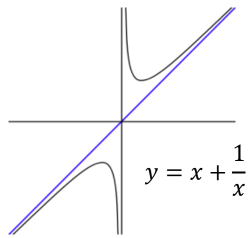


Thomae's function:

$$f(x) = \begin{cases} 1 \text{ if } x = 0 \\ 1/q \text{ if } x = p/q \text{ in reduced form} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

f is continuous in $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous in $\mathbb{Q}$.

Asymptotic lines of a curve are defined with limits. As the asymptote approaches infinity the distance between the line and the curve approaches zero. Asymptotes are horizontal, vertical or oblique. They can also be defined as lines tangent to a curve at infinity.



A proper understanding of limits would resolve many of Zeno's paradoxes, at least for a mathematician even though a philosopher might disagree. A conundrum that might have interested Zeno is whether there is a continuous path that crosses every point in a square or in mathematical term is there a continuous surjective function from the unit interval onto the unit square. The answer is yes. They are called space-filling curves. In 1890 Giuseppe Piano became the first to discover such a curve.

Another space-filling curve is the Hilbert curve, constructed as the limit of a sequence of curves $(H_n)_{n=1}^{\infty}$ where each curve is a mapping from $[0,1]$ into $[0,1]^2$ and a simple modification of the previous curve. The length of $H_n(t)$ is $2^n - 1/2^n$ which makes the length of $H(t)$ infinite.

$$H(t) = \lim_{n \to \infty} H_n(t)$$

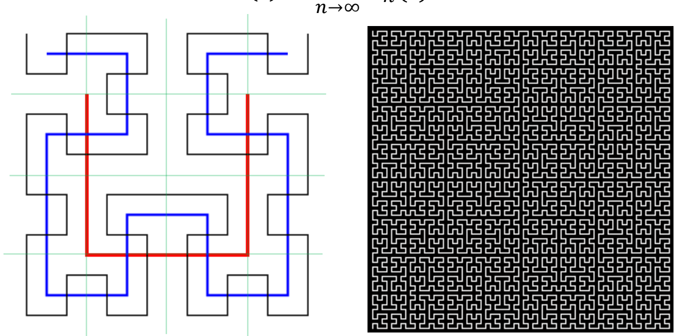
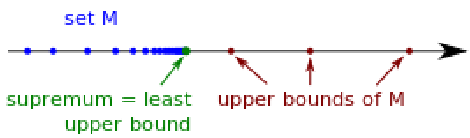


Fig 3.7.1 Iterates of the Hilbert curve H, first H_1, H_2 and H_3 then H_6.

Before we enter into derivation it is time for a closer look at the connection between decimal expansion and real numbers and between real numbers and continuous functions.

An **upper bound** of a set S that is a subset of a partially ordered set $(P, \leq)$ is an element $u \in P$ s.t. $u \geq s$ for every $s \in S$. Lower bounds are defined similarly. A defining property of the real numbers known as the completeness axiom or the least-upper-bound property (LUB).



Definition.
A set X has the LUB property iff efery subset of X with an upper bound has a least upper bound in X (smaller than any other upper bound). This element is called the **supremum** of X, $\sup(X)$. Greatest-lower-bound (GLB) and **infimum** of X, $\inf(X)$ are defined in a similar manner.

Every limited subset of $\mathbb{R}$ has both infimum and supremum but $\mathbb{Q}$ does not, $\{x \in \mathbb{Q} | x^2 < 2\}$ has no supremum in $\mathbb{Q}$. $\mathbb{R}$ is complete (every Cauchy sequence has a limit) but $\mathbb{Q}$ is full of gaps. Useful notations for supremum and infimum are:

$$\sup_{x \in M} f(x) \equiv \sup\{f(x) | x \in M\}, \inf_{x \in M} f(x) \equiv \inf\{f(x) | x \in M\}, \sup_{n \in \mathbb{Z}^+} a_n \dots$$

The supremum s of a set M is characterized by: (ϵ-characterization of sup.)
I: $x \in M \Longrightarrow x \leq s$
II: $\forall \epsilon > 0 \, \exists y \in M : y > s - \epsilon$

This connects nicely to the (ϵ, δ)-definition of limits. From the completeness axiom of $\mathbb{R}$ it follows that every sequence that is monotonic and and limited has a unique limit $\lim_{n \to \infty} a_n = \sup_{n \in \mathbb{Z}^+} a_n$. The expression $0. x_1 x_2 \dots$ is thus a welldefined real number. It is the limit of that corresponds to the limited and increasing sequence $a_n = \sum_{k=1}^{n} x_k \cdot 10^{-k}$.

Conversely every $x = [0,1)$ corresponds to a decimal expression $0, x_1 x_2 \dots$ where the numbers x_k are given by looking at which one of 10 subintervals of decreasing width that x belongs to, starting from $\left[0, \frac{1}{10}\right), \dots, \left[\frac{9}{10}, 1\right)$.

$$\sum_{k=1}^{n} x_k 10^{-k} \leq x < \sum_{k=1}^{n} x_k 10^{-k} + 10^{-n} \text{ for } n = 1,2,3, \dots \rightarrow$$

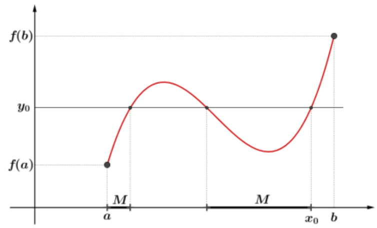
$$\lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{x_k}{10^k} \leq x \leq \lim_{n \to \infty} \sum_{k=1}^{\infty} \frac{x_k}{10^k} + \lim_{n \to \infty} \frac{1}{10^n}$$

The last limit is zero so x corresponds to the sequence $0. x_1 x_2 \dots$

Continuity and and completeness gives some useful properties for functions defined on intervals.

Theorem. (Intermediate value theorem)
Assume $f \in C([a,b],\mathbb{R})$. Then
$y_0 \in [f(a),f(b)] \cup [f(b),f(a)] \Longrightarrow \exists x_0 \in [a,b] : f(x_0) = y_0$

Or in plain English, continuous functions adopt all intermediate values.



Proof.
Assume $f(a) < y_0 < f(b)$ and
define $M = \{x \in [a,b] | f(x) \leq y_0\}$
M limited $\Longrightarrow \exists x_0 = \sup(M) \in [a,b]$

Assume $f(x_0) < y_0$
$f(b) > y_0 \Longrightarrow x_0 < b$
$\lim_{x\to x_0^+} f(x) = f(x_0) \Longrightarrow$ there are points in $[x_0, x_0+\epsilon]$ s.t. $f(x) < y_0$
i.e. points belonging to M which contradicts $x_0 = \sup(M)$
Assume $f(x_0) > y_0$
$f(a) < y_0 \Longrightarrow x_0 > a$
$\lim_{x\to x_0^-} f(x) = f(x_0) \Longrightarrow$ there are points in $[x_0-\epsilon, x_0]$ s.t. $f(x) > y_0$
$[x_0-\epsilon, x_0]$ not in M contradicts $x_0 = \sup(M)$

Since neither $f(x_0) < y_0$ and $f(x_0) > y_0$ we get $f(x_0) = y_0$. ■

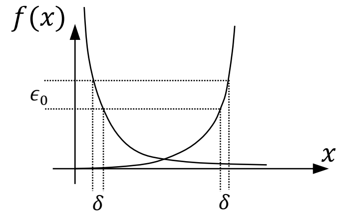
A direct consequence of the theorem is that if f is continuous on an interval I then $f(I)$ is also an interval. $f(x) = x^n$ with $n \in \mathbb{Z}^+$ with $D_f = [0,\infty)$ has $V_f = [0,\infty)$ and polynomials P of odd degree has $V_P = (-\infty,\infty)$ so that polynomial equations like $\sum_{k=0}^{2n+1} a_k x^k = y_0$ has at least one solution.

Continuous functions maps intervals to intervals and compact intervals to compact intervals. A **compact interval** is an interval $[a,b]$ that is limited and closed. To prove this a new version of continuity is needed.

Definition. (Uniform continuity)
A function f, defined on an interval I is uniformly continuous on I if for every $\epsilon > 0$ there is a $\delta > 0$ such that:

$$x,y \in I \;\wedge\; |x-y| < \delta \Longrightarrow |f(x)-f(y)| < \epsilon$$

Continuity of a function on an interval is dependent on a collection of local properties, each point (and its local neighborhood) is considered separately ... $\forall x \exists \delta$ .... There is one $\delta(\epsilon)$ for each $x \in I$ while uniform continuity on an interval is one global property that refers to all pair of points in the interval at the same time ... $\exists \delta \forall x \forall y$ .... There is just one $\delta(\epsilon)$ that handles all $x, y \in I$.



$f(x) = x^{-1}$ is continuous on (0,1] but not uniformly continuous since there is no guarantee to get $|f(x) - f(y)| < \epsilon_0$ no matter how small δ is if you pick x and y with $|x - y| < \delta$ in a subinterval close enough to zero. The same goes for $f(x) = x^2$ on $[0, \infty)$ when $x, y \to \infty$. If however the interval is compact, both closed and bounded then continuity implies uniform continuity. This statement is left to the reader as an exercise.

The importance of compact sets comes from patching up local properties into one global property. A compact set M in a metric space is a set that is both bounded $\sup\{d(x,y)|x,y \in M\} < \infty$ and which contains all its limit points. Compact sets can also be used on spaces consisting of functions with suitably defined distances to prove existence of functions with certain properties. The notion of compactness with the property of turning local properties into global properties can also be generalized into topological spaces.

Theorem. (Continuous functions maps compacts sets into compact sets)
$f \in C(D_f, \mathbb{R}) \wedge [a,b] \subseteq D_f \Longrightarrow f([a,b])$ is a compact interval.

Proof.
I. Boundedness of $f([a,b])$
f is uniformly continuous on $[a,b]$ (exercise).
$\epsilon = 1 \Rightarrow \exists\delta > 0$ s.t. $x, y \in [a,b] \wedge |x - y| < \delta \Rightarrow |f(x) - f(y)| < 1$ $(*)$
Let $r = \frac{\delta}{2}$ and split $[a,b]$ into intervals of length $\leq r$, $a = x_0, x_1, \dots, x_n = b$
$x \in [x_{i-1}, x_i] \Rightarrow |x - x_i| \leq r < \delta \overset{*}{\Rightarrow} |f(x) - f(x_i)| < 1 \Rightarrow$
$|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < 1 + |f(x_i)|$
$M = \max\limits_{i \in \{1,\dots,n\}} (|f(x_i)|) \Rightarrow |f(x)| < 1 + M$ for $x \in [a,b]$

II. Closedness of $f([a,b])$
LUB-property of $\mathbb{R}$ and $f([a,b])$ bounded $\Rightarrow \exists G \in \mathbb{R} : G = \sup(f([a,b]))$
Assume non-existence of $x \in [a,b]$ s.t. $f(x) = G$ and form $(x) = \frac{1}{G - f(x)}$.
g continuous on $[a,b] \Rightarrow g$ limited on $[a,b]$ but
$G = \sup\limits_{x \in [a,b]} f(x) \Rightarrow g$ not limited upwards on $[a,b]$, contradiction so
$f(x) \neq G$ for $x \in [a,b]$ is false $\Rightarrow \exists x_2 \in [a,b]: f(x_2) = G$

Existence of $x_1 \in [a,b]$ s.t. $f(x_1) = \inf(f([a,b]))$ is shown similarly.

$f([a,b]) \subseteq [f(x_1) = y_{\min}, f(x_2) = y_{\max}]$ where $x_1, x_2 \in [a,b]$

f assumes intermediate values $\Longrightarrow f([a,b]) = [y_{\min}, y_{\max}]$ ■

Continuous functions from a compact space to the real numbers attain their maximum and minimum values a.k.a the extreme value theorem.

3.7.2 Derivation

Calculus divides broadly into differential calculus and integral calculus. The close connection between these areas was independently discovered by Leibniz and Newton. They were the two great pioneers of calculus but their work didn’t start from an empty slate. There were predecessors and contemporaries that made their discoveries possible.

Early signs of derivation and integration can be seen in Greek geometry with tangents as a limiting or exceptional case of secants and Archimedes’ calculations of areas and volumes. A more systematic treatment took place after Descartes’ introduction of the coordinate system in 1637, but he was not alone to invent the coordinate system. His rival Fermat had done the same thing. Fermat is considered to be the better mathematician of the two; he also formulated methods for finding maxima, minima and tangents.

Other pioneers of calculus are Roberval and Cavalieri. They independently discovered the method of indivisibles, an early method of integration. It takes a great mind to make a great discovery but when the preparatory work has been made it is only a matter of time before someone breaks through.

A rigorous treatment of analysis based on the definition of limits in the previous section would not come until the 19th century. It was made by Weierstrass, known as the “father of modern analysis”. To sum it all up:

- René Decartes (1596–1650) French.
  Introduced coordinate system and originator of western philosophy.

- Pierre de Fermat (1607–1665) French.
  Coinvented coordinates and a method for finding maxima and tangents.

- Bonaventura Cavalieri (1598–1647) Italian.
  Developed a method for integration and stated Cavalieri’s principle.

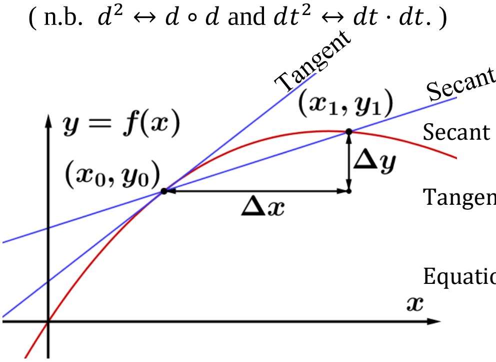
- Gilles de Roberval (1602–1675) French.
  Coinvented Cavalieris methods and connected tangents with motion.

- Gottfried Wilhelm von Leibniz (1646–1716) German.
  Developed differential and integral calculus and notation for it.

- Sir Isaac Neewton (1642–1726) English.
  Calculus and classical mechanics with laws of motion and gravitation.

- Karl Weierstrass (1815–1897) German.
  Rigourus foundation of analysis based on formal definition of limits.

The corner-stone of differential calculus is the derivative. It measures how much the dependent variable (y) changes under variation of the independent variable (x). Their difference quotient will approach the slope of the tangent in the limit when a secant becomes tangential. When the independent variable is time it means instanteneous rate of change. This concept is immensely important in physics where concepts such as speed and acceleration are defined by derivatives. The average speed in a time interval from t_0 to t_1 correspons to the slope of the secant $v = (s_1 - s_0)/(t_1 - t_0) = \Delta s/\Delta t$ and as $t_1 \to t_0$ the instanteneous speed becomes $v = ds/dt$ (m/s) and the acceleration is $a = dv/dt = d^2s/dt^2$ (m/s^2).
(n.b. $d^2 \leftrightarrow d \circ d$ and $dt^2 \leftrightarrow dt \cdot dt$.)



Secant slope: $k_s = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y}{\Delta x}$

Tangent slope: $\mathrm{k_t} = \lim\limits_{x_1 \to x_0} \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{dy}{dx}$

Equation for straight line: $y - y_0 = k(x - x_0)$

In the following we will assume that a function $f: D_f \subseteq \mathbb{R} \to \mathbb{R}$ is defined in a neighborhood of x_0 so that difference quotients are defined when needed.

Definition.
If the following limit exists

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Then the defivative of f at x_0 exists and equals that limit.

Left- and right-sided derivatives are defined by using one-sided limits. A function whose domain D_f is a union of intervals is differentiable if the derivative exists in every point of its domain. This function called the derivative of f may have its own derivative and so forth. There are several notations for derivatives of first and higher orders:

Lagrange:	$f', f'', \ldots, f^{(n)}$	$f'(x_0), \ldots$	
Euler:	$Df, D^2f, \ldots, D^nf$	$Df(x_0), \ldots$	
Newton:	$\dot{y}, \ddot{y}, \dddot{y}$ (dot notation)	$\dot{y}(t_0), \ldots$	(often with time)
Leibniz:	$\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \ldots, \dfrac{d^ny}{dx^n}$	$\left.\dfrac{dy}{dx}\right\rvert_{x=x_0}$ or $\dfrac{dy}{dx}(x_0)$	

Leibniz notation is based on a repeated use of an infinetsimal version $\Delta \curvearrowright d$ of the forward difference operator $\Delta = E - I$. E is the forward operator from page 108 acting on a sequence. For the sequence $y_\delta = \langle f(a + k \cdot \delta)\rangle_{k=0}^{\infty}$ the third derivative is approximated at the points $(a, a + \delta, a + 2\delta, \ldots)$ by:

$$\frac{\Delta^3 y_\delta}{\delta^3} = \frac{(E-I)^3 y_\delta}{\delta^3} = \langle \frac{f(a+3\delta) - 3f(a+2\delta) + 3f(a+\delta) - f(a)}{\delta^3}, \ldots \rangle$$

$$\Delta(\Delta(\Delta y_\delta/\delta)/\delta)/\delta \rightarrow \langle \frac{d^3 f}{dx^3}(a), \frac{d^3 f}{dx^3}(a+\delta), \frac{d^3 f}{dx^3}(a+2\delta), \ldots \rangle \text{ as } \delta \rightarrow 0$$

A function is said to be continuously differentiable and belong to class C^1 if it is differentiable and the derivative is a continuous function. To be of class C^k requires existence of $f, f', \ldots, f^k$ and f^k should be continuous. Existence of f^k implies continuity of all derivatives $f^{(i)}$ with $i < k$. A function is called smooth and belonging to class C^∞ if f^k exists for every $k \in \mathbb{Z}^+$.

Theorem.
$\mathrm{D}x^n = nx^{n-1}$ for $n \in \mathbb{N}_0$

Proof

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)^n - a^n}{h} =$$

$$\lim_{h \to 0} \sum_{k=0}^{n-1} \binom{n}{k} a^k h^{n-1-k} = na^{n-1} \quad \Rightarrow \quad f'(a) = na^{n-1} \Rightarrow \quad Dx^n = nx^{n-1} \quad \blacksquare$$

$D^k x^n = [k \le n] \cdot n(n-1) \ldots (n-k+1) x^{n-k} = [k \le n] \cdot n^{\underline{k}} x^{n-k}$ with special case $D^n x^n = n!$ and $D^m x^n = 0$ if $m > n$ so $f(x) = x^n \in \mathrm{C}^\infty$.

Derivatives of other functions will be given or derived once they have been properly defined. As for limits there are rules that makes many derivatives easy to calculate: (All given functions are assumed differentiable)

Linearity
$(af + bg)' = af' + bg'$ where $a, b \in \mathbb{R}$

Product rule (Leibniz's law)
$(f \cdot g)' = f' \cdot g + f \cdot g'$

Derivationlike operators occur in many areas of mathematics, with or without limits. Leibniz law is the common property of these operators.

Generalized Leibniz rule

$$\mathrm{D}^k(f_1 f_2 \cdot \ldots \cdot f_n) = \sum_{j_1+j_2+\cdots+j_n=k} \binom{k}{j_1, j_2, \ldots, j_n} \prod_{i=1}^{n} f_i^{(j_i)}$$

Quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$ where g is nonzero

Chain rule

$$\mathrm{D}[f(g(x))] = f'(g(x)) \cdot g'(x) = \frac{d}{dz} f(z)\Big|_{z=g(x)} \cdot \frac{d}{dx} g(x)$$

Inverse function rule

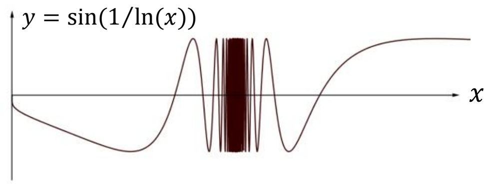
$$f \circ g = \mathrm{id} \wedge g \circ f = \mathrm{id} \rightarrow \quad g' = \frac{1}{f' \circ g} \qquad \left(\frac{dx}{dy} = \frac{1}{dy/dx} \right)$$

Leibniz's notation is often very suggestive. If $y(x) = f(g(h(x)))$ then we only need to know the derivatives of f, g and h to find the derivative of $y(x)$.

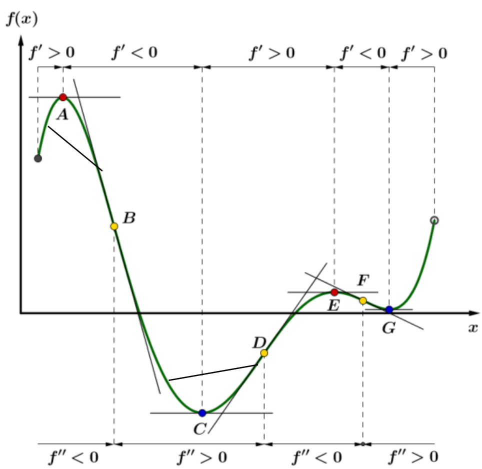
$$\frac{dy}{dx} = \frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dh} \cdot \frac{dh}{dx}$$

with

$$f = \sin g \,, g = 1/h \,, h = \ln x$$



Derivatives of elementary functions like $\sin x$ and $\ln x$ will be given when we have introduced proper definitions. With derivatives and differentiation rules graphs can be analyzed and optimization problems can be solved.



Function is increasing if $f'(x) \geq 0$
Function is decreasing if $f'(x) \leq 0$

Stationary points have $f'(x) = 0$

All line segments are above the graph
the function is convex: $f''(x) \geq 0$
All line segments are below the graph
the function is concave: $f''(x) \leq 0$

Inflection points: $f''(x) = 0$

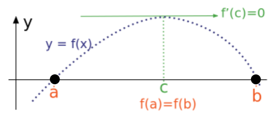
Local minima if $f'(x) = 0 \,, f''(x) > 0$
Local maxima if $f'(x) = 0 \,, f''(x) < 0$

These properties follows easily from definitions of derivation and extrema.

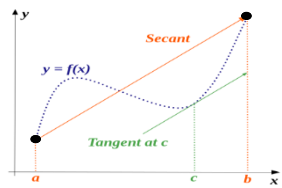
Definition. (Extrema)
A real-valued function f defined on a domain X has a **global maximum** at x^* if $f(x) \leq f(x^*)$ for all $x \in X$ and correspondingly for minimum.
f has a **local maximum** at x^* in a metric space X if $\exists \varepsilon > 0$ s.t. $f(x) \leq f(x^*)$ for all $x \in X$ with $\|x - x^*\| < \varepsilon$ and ditto for minimum.

Another set of useful theorems from calculus are the following:

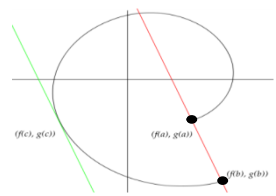
Theorem. (Rolle's theorem, a special case of the mean value theorem.)
If $f: \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$ then
$\exists c \in (a, b) : f'(c) = 0$



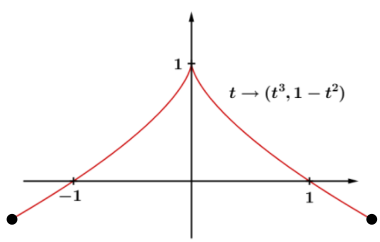
Theorem. (Mean value theorem.)
If $f: \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then
$\exists c \in (a, b) : f'(c) = \frac{f(b)-f(a)}{b-a}$



Theorem. (Generalized mean value theorem.)
If $f, g: \mathbb{R} \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) then $\exists c \in (a, b)$ s.t.
$\big(f(b) - f(a)\big)g'(c) = \big(g(b) - g(a)\big)f'(c)$



Existence of a tangent parallel to the secant in the last case is not always guaranteed. If $f'(c) = g'(c) = 0$ then a tangent might not even be defined at c. This is the case for $t \mapsto (t^3, 1 - t^2)$ which has a **cusp**, a place on a graph $t \mapsto (f(t), g(t))$ where both f' and g' are zero and at least one of them changes sign.



Proof. (Rolle's theorem)
If f is constant on $[a, b]$ then $f' \equiv 0$ on (a, b). If not then there must be an interior point $c \in (a, b)$ where f adopts an extremum for $[a, b]$ (see p.172).
$\frac{f(x)-f(c)}{x-c}$ will have different signs on (a, c) and (c, b) and since the limit when $x \to c$ exists it must be zero, i.e. $f'(c) = 0$. ∎

Applying Rolle's theorem to $g(x) = f(x) - \frac{f(b)-f(a)}{b-a} \cdot x$ with $g(a) = g(b)$ gives the mean value theorem for $f(x)$.

When Leibniz introduced the notation dy/dx he thought ot it as a division between an infinetisimally small change in the y-value of a function $y(x)$ in response to an infinitesimally small change in the x-value. These infinitesimals dy and dx where not real numbers but their quotient would become a real number. This informal and confusing mixing of different entities was later cleared up by a more rigorous treatment in terms of limits.

An alternative way of introdicing rigor is given by non-standard analysis where infinitesimal numbers are introduced in a strict way with well-defined properties and relations to real numbers. The informal use of infinitesimals is still used in physics. Textbooks on thermodynamics and treatments of error estimations are full of formulas with infinitesimals. The best way to look at these infinitesimals are as differentials.

Definition. (Differential)
The differential of $f(x): \mathbb{R} \to \mathbb{R}$ is a function df of x and the increment Δx.

$$df(x, \Delta x) \stackrel{\text{def}}{=} f'(x) \cdot \Delta x$$

The variables on the left are often omitted and with $f(x) = x$ we get the differential $dx = \Delta x$. The differential of $f(x)$ becomes $df = f'(x)dx$.

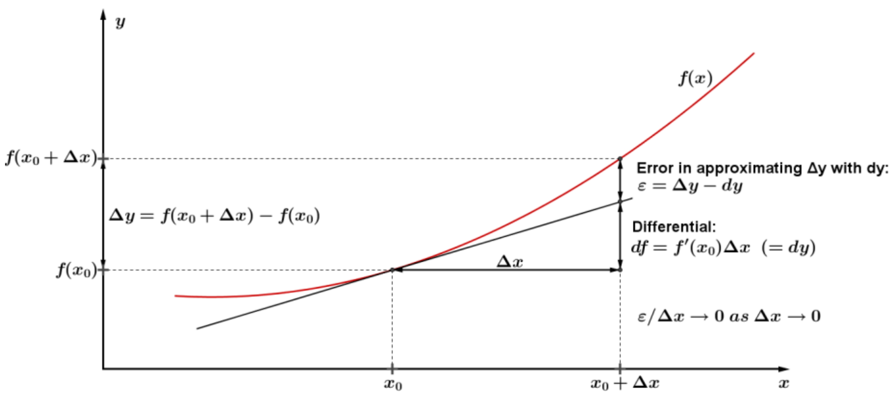


fig. 3.7.2 Function $f(x)$ and its differential $df(x, \Delta x) = f'(x)\Delta x$.

The differential is the best linear approximation of the increment in y as a function of the increment in x. It's known as the principal or linear part of the increment of a function.

Derivatives and differentials have natural generalizations for functions of several variables with with partial derivatives and total differentials.

Functions of several variables $f(x, y, z, \ldots)$ or $f(x_1, x_2, \ldots, x_n), n > 1$ have partial derivatives where one variable varies and the others are kept fixed. The partial derivative of an n-ary function $f(x_1, \ldots, x_n)$ in the direction x_i at the point $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ is defined as:

$$\frac{\partial f}{\partial x_i}(\boldsymbol{a}) = \lim_{h \to 0} \frac{f(a_1, \ldots, a_i + h, , \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

(all variables except x_i are fixed)

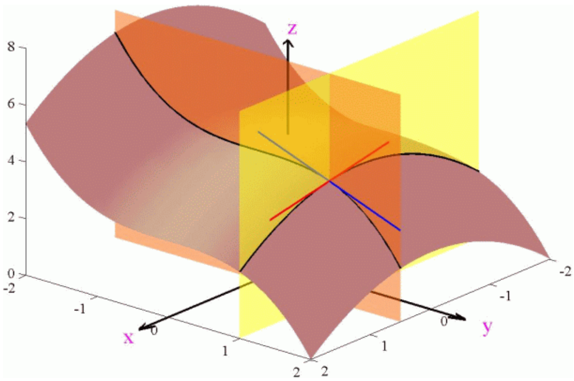


Fig. 3.7.3 Partial derivatives $\partial f / \partial x\,(\boldsymbol{a})$ and $\partial f / \partial y\,(\boldsymbol{a})$ of $f(x, y)$.

As $\boldsymbol{a}$ is varied $\partial f / \partial x_i$ becomes a function over the domain of f for which new derivatives can be taken. Some notations for 1st and 2nd orders are:

$$f'_x = f_x = \partial_x f = D_x f = D_1 f = \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x} = f_x(x, y, \ldots) = \frac{\partial f}{\partial x_1}(x_1, \ldots, x_n)$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = \partial_{xx} f \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy} = \partial_{yx} f = f_{xy}$$

$\partial_{yx} f = \partial_{xy} f$ if the second derivatives are continuous. When all the partial derivatives up to order n are continuous, then the order for partial derivation will not matter and the multi index notation for partial derivation can be used:

$$\begin{matrix} \alpha = (\alpha_1, \ldots, \alpha_n) \\ |\alpha| = \alpha_1 + \cdots + \alpha_n \end{matrix} \quad \alpha_i \in \mathbb{N}_0 \quad \begin{matrix} \text{Partial derivative} \\ \text{of order } |\alpha| \end{matrix} \quad \frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

The differential for a fuction $f: \mathbb{R}^n \to \mathbb{R}$ at $\boldsymbol{x}_0 = (x_1, x_2, \ldots, x_n)$, i.e. the best linear approximation of f around $\boldsymbol{x}_0$ is:

$$df(\boldsymbol{x_0}, \Delta \boldsymbol{x}) = \partial_1 f(\boldsymbol{x}_0)\Delta x_1 + \partial_1 \boldsymbol{f}(\boldsymbol{x}_0)\Delta x_1 + \cdots + \partial_n f(\boldsymbol{x}_0)\Delta x_n$$

Further excursions into the realm of multivariable analysis will have to wait for a later part of our journey.

3.7.3 Integration

If derivation is the first leg of calculus then integration is the second leg of calculus. Derivation deals with slopes and integration deals with areas. The fundamental theorem of calculus brings these seemingly different operators together. Rudimentary forms of the theorem were given before Leibniz and Newton but it was they who really integrated diffrential calculus and integral calculus into one area.

In Leibniz notation derivation is a quotient of two infinitesimals dy/dx and integration is an infinite sum of infinitesimals $\sum_{i=0}^{\infty} y_i dx$ which can be used to calculate the area under a graph $y(x)$. Areas below the x-axis are negative.

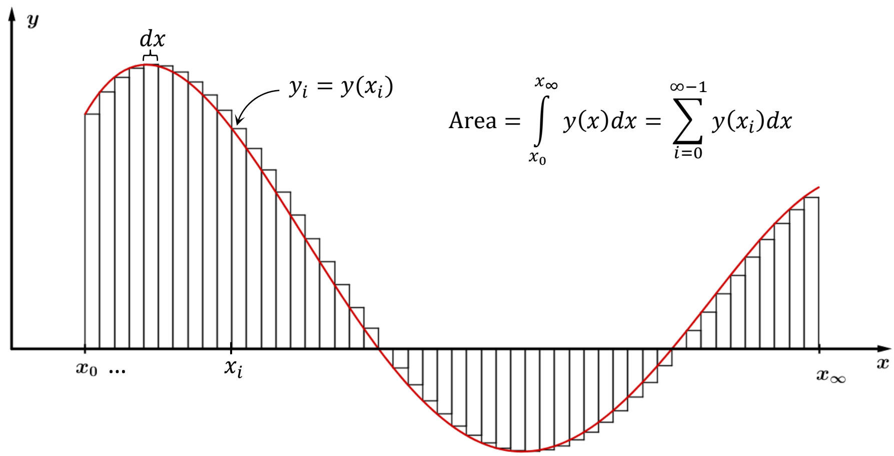


fig. 3.7.4 Area as an infinite sum of infinitesimal rectangular stripe areas.

The fundamental theorem of calculus is quite obvious for the derivative of a function. The sum of differential increments equals the total difference.

$$\int_{x_0}^{x_\infty} \frac{dy}{dx} dx = \sum_{i=0}^{\infty-1} \frac{dy}{dx}(x_i)\, dx = \sum_{i=0}^{\infty-1} y(x_{i+1}) - y(x_i) = y(x_\infty) - y(x_0)$$

Definition. (Antiderivative)
An **antiderivative** of a function $f(x)$ also known as a **primitive function**, primitive integral or indefinite integral is a function $F(x)$ s.t. $F'(x) = f(x)$, an alternative notation for $F(x)$ is $\int f(x)dx$, **undefinite integral**.

Theorem. (Fundamental theorem of calculus)

$$\int_a^b f(x)\, dx = F(b) - F(a) \equiv [F(x)]_a^b$$

Gottfried Wilhelm Leibniz (1646–1716)

Cox’s IQ-ranking of historic persons put Leibniz in second place after Goethe. This is not to be taking too seriously; Goethe never produced anything of enduring scientific value. Leibniz was a polymath, active at the highest level in many disciplines. He made important contributions in philosophy, history, theology, biology, medicine, psychology (consciousness and perception), geology (earth molten core), politics, law, technology and more, a true “Renaissance man” comparable to Da Vinci or Galileo. Leibniz had no false modesty; he described himself as “the most teachable of mortals”.

Leibniz was born in Leipzig at the end of thirty years of war in the Thirty Year’s War (1618–1648) fought between Catholic rulers and rulers of the reform movement. The wars had left German parts of Europe devastated. France became the dominant power and the long reign of Louis XIV (1643–1715) would last Leibniz entire lifespan.

Leibniz grew up in a pious Lutheran family. His father was a professor of moral philosophy that died when Gottfried was only six years old. He had a big personal library, full of Latin texts on theology and philosophy. The young Gottfried quickly learned Latin and started reading. At age 15 he entered the university to study philosophy and law. His application in 1666 for a doctorate in Leipzig was turned down in spite of his known abilities, probably due to his young age. The following year he earned a license to practice law and a doctor’s degree in a university outside Nuremberg.

Leibniz first job was as a secretary in an alchemical society before he was employed as an assistant to Baron von Boineburg, who served the local ruler in Mainz. Leibniz came to work with a redraft of the legal code for the Elector of Mainz and got him interested in a plan to protect German areas from French intervention. The plan was to persuade Louis XIV to engage elsewhere, in a war with Egypt as a stepping stone for a conquest of the Dutch East Indies. The plan became irrelevant with the onset of the Franco-Dutch war 1672–1678 but it started Leibniz’s diplomatic career.

In 1672 the French government invited Leibniz to Paris where he met Christiaan Huygens, a leading scientist of the time. Leibniz realized his shortcomings. Huygens became his mentor in mathematics and physics.

The following year Leibniz was sent on a mission to the English government. In England he presented his mechanical calculating machine for the Royal Society. He had been working on it for some years and it was the first calculator that could handle elementary arithmetic with $+, -, \times, \div$. The mission ended with news of the Elector's death and Leibniz needed a new job. He yearned for the intellectual climate in Paris or possibly the Habsburg imperial court but had to settle for a position with the Duchy of Brunswick-Lüneburg, home of the House of Hanover dynasty.

Leibniz worked for the Hanover dynasty 1676–1716 as a political adviser, librarian and historian. He was commissioned to write a book on the history of the House of Hanover which he did but in a much more thorough and time consuming way than his employers had hoped. It was not published until the 19th century.

Although Leibniz never married he corresponded with several influential women in the upper classes. Among his friends, benefactors and students where Electress Sophia of Hanover, her daughter Sophia, queen of Prussia and Caroline of Ansbach, future queen in England.

From 1712 Leibniz spent two years at the Habsburg court where he took an active part in setting up his patron Sophia for the British throne. There were 50 persons closer in line but they were all Catholics and excluded by the Act of Settlement from 1701. Sophia died a few months before queen Anne but her son became king George I of Great Britain. Leibniz hoped to gain a position at the English court but Gottfried and George were not the best of friends. George was irritated with Leibniz for constant delays with the historical chronicle and even worse was the bitter conflict with Newton over the invention of calculus. The dispute started in 1708 and would come to cloud the rest of Leibniz' life.

When Leibniz died in 1716 he was out of favor with important people, the rational school of philosophy that he belonged to was eclipsed by empiricism. Much of his work was contained in private communications and his reputation was in decline. His grave went unmarked for 50 years even though he had become a member of the academies of science in both Paris and London. Leibniz always emphasized the collaborative endeavor of science. He founded the Berlin Academy of Sciences and he became its first president from 1700 to his death. In 1711 he was visited by Tsar Peter the Great of Russia. After that he took great interest in Russian affairs and together they prepared for the St. Petersburg Academy.

Leibniz was given the opportunity to combine his work at different courts for various rulers with his own pursuit of various projects. Much of this work was done in his vast correspondence. There were more than 600 correspondents, many of them leading scholars in their field. It was a lifelong ambition of Leibniz to assemble all human knowledge.

Leibniz pursuit of knowledge started with his mother who had a large influence on him, especially in moral and religious questions. He was a Lutheran but with a lifelong aim to unify Catholic and Lutheran churches. He argued for a European confederation with a uniform Christianity.

In one of his most important publications the *Theodicé* from 1710 he sat down the principles of divine justice and he claimed that the world was the best of all possible worlds despite apparent suffering, injustice and natural disasters. He argued from assuming that God was a perfect being and if it had been possible for God to create a better world then he would have done so.

Leibniz' belief was later satirized by Voltaire in *Candide* where Leibniz is portrayed as doctor Pangloss, a character that keeps claiming that we live in the best of all worlds even after a long series of personal disasters and human tragedies. In the end after many calamities his protégée the young Candide stops believing in him, starts thinking independently and takes a more realistic view of the world and of religion.

Leibniz saw no contradiction between reason and faith. Any part of religion not in line with reason must be discarded from a true understanding of religion. When it comes to philosophy Leibniz belonged to the rational school of continental Europe. They claimed true knowledge comes from applying reason to first principles in contrast to the empirical school strong in Britain. Rationalism was in some ways a continuation of the scholastic tradition but without theology. A famous saying attributed to Leibniz is: Calculemus "Let us calculate to see who is right".

Part of his rational philosophy was a belief in a universal language of logic with its own calculus to decide correct from incorrect reasoning. His ideas on logic remained unpublished and it took 200 years before the ideas resurfaced in modern formal logic with George Boole and others. In many ways he was a forerunner to modern computer science. He refined the binary number system and improved Pascal's calculating machine to handle multiplication and division.

The calculator was not his only contribution to applied science and mechanical devices. Among his other designs were lamps, clocks, windmills, water pumps, hydraulic presses and submarines. Combining theory with practice was his motto. Leibniz' work ranged from very practical to highly speculative.

Newton contributed more to the development of physics and mechanics than Leibniz but in the end it was Leibniz' idea that replaced Descarte's and Newton's idea of space and time that existed independently of matter as an empty container. Leibniz had a relational view of space/time and matter that was more in line with the theories of Mach and Einstein that extended and replaced Newton's mechanics and gravitational theory.

Leibniz mathematical career started late. He had begun to study motion and constructing a calculating machine around 1670 and was seeking contacts with scientists. With the trips to Paris and London in 1672–1673 he got connections and realized that he needed to learn more if he was going to achieve something that matched his ambitions. He began working on the geometry of infinitesimals and struggled to find a good notation.

Leibniz notation for derivation dy/dx and integration $\int f(x)dx$ where d means infinitesimal difference and the elongated S comes from summa are now standard. Notation was important to Leibniz. He pointed out that analysis was already known in ancient Greece and his contribution was the notation which "express the exact nature of a thing briefly … the labour of thought is wonderfully diminished".

By 1676 he had derived much of what is taught in calculus in high school, such as $(fg)' = f'g + fg'$ and $D(x^{\alpha}) = \alpha x^{\alpha-1}$ for $\alpha \in \mathbb{Q}$. The dispute between Leibniz and Newton over precedence was rooted in the long time between thought and publication. Leibniz work on differential calulus was published in 1684 and his results on integral calculus were printed in 1686. Newton's *Principa* appeared the following year but he had written *Method of Fluxions* already in 1671 but he failed to get it published.

When Leibniz published in 1684 the notation was new and he gave no proofs. Jacob Bernoulli called it an enigma rather than an explanation. Proper proofs based on Leibniz characterization of infinetismal numbers were later implemented in non-standard analysis, a formal and rigorous extension of the real number system.

Among Leibniz other contributions to mathematics are:

- Arranging coefficients of a system of linear equation into a matrix.
- Solving these equations by means of Gaussian elimination.
- The concept of a matrix determinant calculated by Leibniz formula.

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

- The terms function, variable, parameter and coordinate.
- Notation for the the n-th root $\sqrt[n]{x}$.
- The chain rule: $(f \circ g)' = (f' \circ g) \cdot g'$
- $\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f\big(x, b(x)\big) \cdot \frac{db(x)}{dx} - f\big(x, a(x)\big) \cdot \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f(x,t)}{\partial t} dt$
- Leibniz formula for pi: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots$

The last formula is a special case of the serial expansion of $\arctan(x)$.

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

This formula was however already known 300 years earlier by Madhava an Indian mathematician and astronomer who discovered the infinte series for the trigonometric functions of sine, cosine, tangent and arctangent.

300 years after Leibniz presented his first calculating machine to the royal society in London in 1673 came the first pocket calculators. Leibniz made three more machines based on his original model. None of these machines has survived and in 1690 he made a final version “machina arithmetica”. This too fell into oblivion but it was rediscovered in 1894 in an attic of the University Church of Göttingen. The machine can now be seen at the Gottfried Wilhelm Leibniz Library in Hannover together with the private library of Leibniz.

A proper definition of integration based on limits and not on infinitesimals was given by Berhard Riemann in 1854.

Definition. (Partition)
A partition Δ of an interval $[a,b]$ is a finite sequence,
$\Delta = \{x_0, x_1, \dots, x_n\}$ s.t. $a = x_0 < x_1 < x_2 < \cdots < x_n = b$
The norm of a partition equals the longest subinterval.
$\|\Delta\| = \max_{0 \le i \le n-1} (x_{i+1} - x_i)$

Definition. (Riemann integral)
Let $f: [a,b] \to \mathbb{R}$ then the integral $\int_a^b f(x)dx$ exists and equals S iff:
$\forall \varepsilon > 0 \exists \delta > 0$ s.t. for any partition $\Delta = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ with $\|\Delta\| < \delta$
$\left| \underbrace{\sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)}_{\text{Riemann sum}} - S \right| < \varepsilon$ for any choice of t_i where $t_i \in [x_i, x_{i+1}]$.

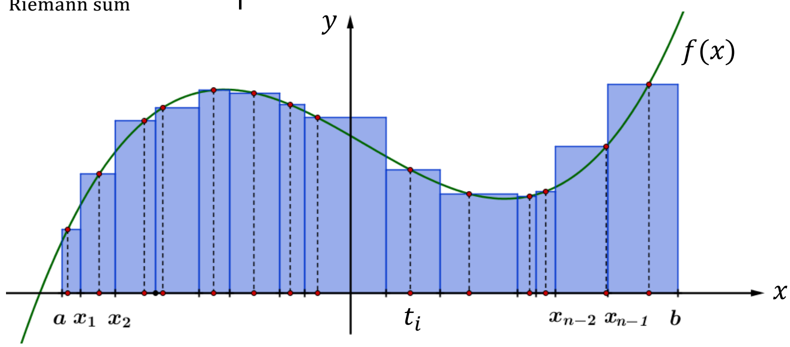


Fig. 3.7.5 Riemann sum for the integral $\int_a^b f(x)dx$.

There is an alternative definition of the Riemann integral that can be shown to be equivalent to the one given above. It is given in terms of upper $\bar{I}$ and lower $\underline{I}$ estimates of $I = \int_a^b f(x)dx$. These are called Darboux integrals.

For a partition $\Delta = \{x_0, x_1, \dots, x_n\}$ where $\Delta x_k \equiv x_k - x_{k-1}$ let

$$M_k = \sup_{x_{k-1} \le x \le x_k} f(x) \text{ and } \bar{S}(\Delta) = \sum_{k=1}^{n} M_k \Delta x_k \text{ and } \bar{I} = \sup_{\Delta} \bar{S}(\Delta)$$

$$m_k = \inf_{x_{k-1} \le x \le x_k} f(x) \text{ and } \underline{S}(\Delta) = \sum_{k=1}^{n} m_k \Delta x_k \text{ and } \underline{I} = \inf_{\Delta} \underline{S}(\Delta)$$

If $\bar{I} = \underline{I}$ then f is Riemann integrable and $\int_a^b f(x)dx = \underline{I} = \bar{I}$. With this definition it's easy to find a non integrable function.

Definition.
The **characteristic function** also known as the **indicator function** of a subset A of a set X is the function $I_A: X \to \{0,1\}$ defined as:
$I_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$ or with Iverson bracket notation $I_A(x) = [x \in A]$.

The characteristic function of the rationals $I_{\mathbb{Q}}: \mathbb{R} \to \{0,1\}$ can't be integrated on $[0,1]$ since the upper and lower Darboux integrals do not coincide, $\underline{I}= 0$ and $\overline{I} = 1$. There is however a more general way to define integration that can handle even the case $\int_0^1 I_{\mathbb{Q}}(x)dx$. It is called Lebesgue integration. A function bounded on $[a, b]$ has a Riemann integral iff it is continuous almost everywhere which means outside a set of Lebesgue measure zero.

The antiderivative of $f \in C[a, b]$ with $F(x_0) = y_0$ can be defined by $(x) = \int_{x_0}^{x} f(t)dt + y_0$. The general antiderivative is undecided up to an additive constant usually denoted by C and called the integration constant. A variable like t above that can be freely exchanged for another letter is sometimes called a "dummy variable". It is a placeholder or a **bound variable** whereas x above has an effect on the value of the expression, making it a **free variable**. Other names and notations for antiderivative are primitive integral, indefinite integral, $D^{-1}f(x)$, $\int f dx$ and $\int f(u)du$.

Fundamental theorem of calculus
Let $f: [a, b] \to \mathbb{R}$ be a continuous function.
Define $F: [a, b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)dt$.
This is possible since f will be uniformly contiuous on $[a, b]$ and then for each $\varepsilon > 0$ there will be a $\delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}$ and for every partition with $||\Delta|| < \delta : \overline{S} - \underline{S} < \sum_k \frac{\varepsilon}{b-a} \Delta x_k = \varepsilon \Rightarrow \underline{I} = \overline{I}$.

Before looking at the derivative of $F(x)$ we need the mean value theorem for definite integrals which says: $\frac{1}{b-a} \int_a^b f(x)dx = f(c)$ for some $c \in [a, b]$.
This follows from the theorem of page 172, $f([a, b]) = [m, M]$ that implies:

$$m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M \Rightarrow \exists c \in [a, b]: \int_a^b f(x)dx = f(c)(b - a)$$

With this in place the derivative of $F(x)$ can be proved to exist:

$$F'(x) \equiv \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_x^{x+\Delta x} f(t)dt = \lim_{\Delta x \to 0} f(c)$$

for some $c \in [x, x + \Delta x]$, this can be written as a function $c = c(x, \Delta x)$.

Applying the squeeze theorem on $x \leq c(x, \Delta x) \leq x + \Delta x$ as $\Delta x \to 0$ and the limit theorem for a composition $f(c(x, \Delta x))$ with f continuous gives: $\lim_{\Delta x \to 0} f(c) = f(x)$ which means that $F(x) = \int_a^x f(t)dt$ has $F'(x) = f(x)$.

As a corollary $D\left(F(x) - \int_a^x f(t)dt\right) = 0$ and from the mean value theorem $F(x) - \int_a^x f(t)dt$ must be constant throughout $[a, b]$, call it C.

The general antiderivative of $f(x)$ becomes $F(x) = \int_a^x f(t)dt + C$ and

$$\int_a^b f(t)dt = F(b) - F(a) \qquad \blacksquare$$

With this we have finally integrated derivation that deals with slopes and integration that deals with areas.

Some properties of antiderivatives and definite integrals are:

Linearity

$\int \left(\alpha f(x) + \beta g(x)\right)dx = \alpha \int f(x)dx + \beta \int g(x)dx$

Conactenated intervals

$\int_a^b f(x)dx + \int_b^c f(x)dx) = \int_a^c f(x)dx$

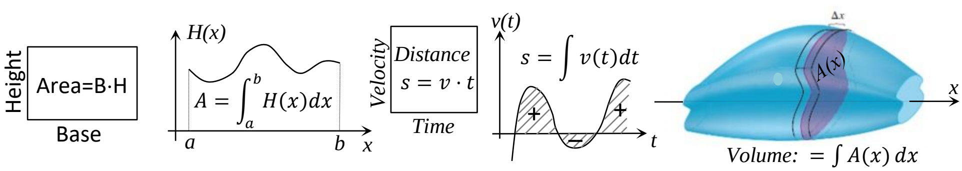
Substitution

$\int_{\varphi(a)}^{\varphi(b)} f(x)dx = \int_a^b f\left(\varphi(t)\right)\varphi'(t)dt \qquad \{\, x = \varphi(t) \text{ and } dx = \varphi' dt \,\}$

Integration by parts

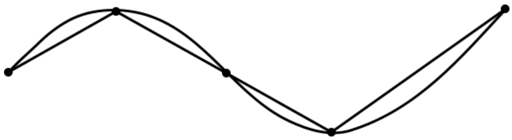
$\int_a^b fg'dx = [fg]_a^b - \int_a^b f'\, gdx \qquad \{\, D^{-1}(fg' + f'g) = fg \,\}$

The illustrations of integration so far have been of area (with sign) but integration is much more general, it is about summing up basic small units that each contribute to a total value. The basic unit can be n-dimensional boxes whose contributions equal their volume but they can also be of other shapes.



The same applies to derivation that generalizes division. In physics the operation of derivation (division) and integration (multiplication) are reflected in the units that are used for measurement, for instance:

Velocity: $v = ds/dt$ (Unit: 1m/s) and Work: $W = \int Fdx$ (Unit: 1N·m)



Calculus provides a unified way of measuring the arc length of a curve. If $[a,b] \ni t \curvearrowright f(t) \in \mathbb{R}^n$ is a parametric representation of a curve in $\mathbb{R}^n$ and $\Delta = \{a = t_0, t_1, \dots, t_n = b\}$ is a partition of $[a,b]$ then the length of the polygonal path $L(\Delta, \boldsymbol{f}) = \sum_{i=1}^{n} \|\boldsymbol{f}(t_i) - \boldsymbol{f}(t_{i-1})\|$ will be a lower bound of the arc length. If there is a largest lower bound the curve is called rectifiable and $\sup_{\Delta} L(\boldsymbol{f}, \Delta)$ is the curve's arc length. If $\boldsymbol{f} \in C^1[a,b]$ then:

$$L(\boldsymbol{f}) = \sup_{\Delta} L(\boldsymbol{f}, \Delta) = \lim_{\|\Delta\| \to 0} L(\boldsymbol{f}, \Delta) = \lim_{n \to \infty} \sum_{i=1}^{n} \left\| \frac{\boldsymbol{f}(t_i) - \boldsymbol{f}(t_{i-1})}{\Delta t} \right\| \Delta t = \int_a^b \|\boldsymbol{f}'(t)\| dt$$

With t as a time parameter, $\boldsymbol{f}'(t)$ will be the velocity vector, $\|\boldsymbol{f}'(t)\|$ is the speed at time t and $L(\boldsymbol{f})$ is the arc length in coordinate units. A change of time unit, reparametrization, will not change the arc length.

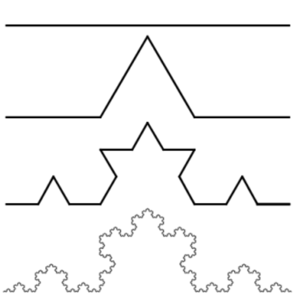
A curve described by a graph $y = g(x)$ with $x \in [a,b]$ has a parametrization $t \curvearrowright \boldsymbol{f}(t) = (t, g(t)) \to \boldsymbol{f}'(t) = \big(1, g'(t)\big) \to \|\boldsymbol{f}'(t)\| = \sqrt{1 + (g'(t)^2}$.

$$\underset{a \le x \le b}{L(g)} = \int_a^b \sqrt{1 + (g'(x))^2} dx = \int_a^b (1 + (dy/dx)^2)^{1/2} dx$$

By definition $\pi \equiv$ Circumference/Diameter of the unit circle $x^2 + y^2 = 1$.

$$g(x) = (1 - x^2)^{1/2} \quad \to \quad \pi = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}$$

Every continuous curve is not rectifiable, curves with no upper bound for their polygonal paths are of infinite length. The **Koch curve** is defined iteratively by replacing the middle third of each line segment with an equilateral outgrowth. It converges to a continuous curve with infinite length.

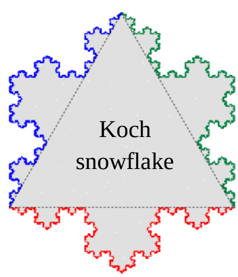


$K_0 : L = 1$

$K_1 : L = 4/3$

$K_2 : L = (4/3)^2$

$K_n : L = (4/3)^n \to \infty$ as n $\to \infty$



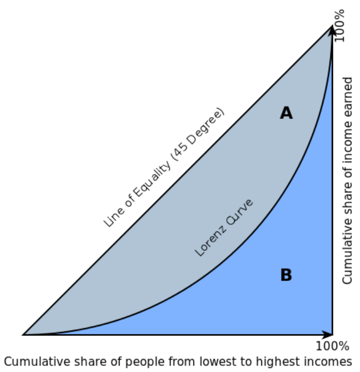
An infinite boundary of a finite area.

It's high time that we looked at some actual integration but for that we need functions and antiderivatives which we have not very many of so far but we do have $Dx^n = nx^{n-1}$ which leads to $D^{-1}x^n = x^{n+1}/(n+1) + C$.

The Gini coefficient

The Gini coefficient G is used to measure spread in the distribution of income or wealth. It is based on the Lorentz function $L: [0,1] \to [0,1]$ with $L(x\%)$ being the needed share of people (measured from below) for their summed income/wealth to equal $x\%$ of the total income of the group. $L(x)$ is an increasing function and assuming no negative income/wealth $L(0)$=0 and $L(1)$=1.

Complete equality (Everybody has the same income/wealth): $L(x) = x$.
Maximal inequality (One person earns/owns everything): $L(x) = [x = 1]$.



$G \equiv A/(A+B) = 2A = 1 - 2B$
A and B are the areas in the diagram.
$0 \leq G \leq 1$

$L(x) = x^n, B = \int_0^1 x^n dx\,, G = \frac{n-1}{n+1}$

Any distribution with the same area A has the same Gini coefficient.

Gini coefficients of income in different countries varies from 0.2 to 0.6 which corresponds to the Gini coefficients of $x^{1.5}$ and x^4. World Gini coefficients has been estimated to be in decline, since 1988 from G=0.8 to G=0.65 or $x^9 \to x^{4.7}$.

Integration ∫ and summation Σ share so many properties besides their use of different versions of S that one wonders if there could be a way to unite them and cover both discrete and continuous versions of a statement with one proof. There is, it's called the **Riemann-Stieltjes integral** and you get it by simply replacing the measure of each interval in a partition. Instead of using the length $\Delta x_k = x_k - x_{k-1}$, use another measure $\alpha(x)$ and let each interval get a weight $\alpha(x_k) - \alpha(x_{k-1})$. It's most natural to let $\alpha(x)$ be an increasing function but it does not have to be.

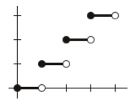
$$\int_a^b f d\alpha = \int_a^b f(x) d\alpha(x) = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(\xi_k)(\alpha(x_k) - \alpha(x_{k-1}))$$

When a measure function $\alpha(x)$ has a discontinuity with a jump at $x_0 \in [a,b]$ it gives an extra contribution to the RS-integral:

$$j = \left(\lim_{x \searrow x_0} \alpha(x) - \lim_{x \nearrow x_0} \alpha(x)\right) \neq 0 \quad \Rightarrow j \cdot f(x_0) \text{ added to } \int_a^b f(x)d\alpha(x)$$

Ordinary summation becomes a special case of RS-integration:

$$\sum_{k=1}^{n} f(k) = \int_0^n f d\alpha \quad \text{With } \alpha(x) = \lfloor x \rfloor$$



This makes the RS-integral extra suited for probability theory where a real-valued random variable X can range over both discrete and continuous parts. The probability distribution of X is given by the **cumulative distribution function** (CDF) with $F_X(x) = P(X \leq x)$ (Probability of $X \leq x$).

$$\mathrm{P}(a < x \leq b) = F_X(b) - F_X(a) \rightarrow \mathrm{P}(a < x \leq b) = \int_a^b dF_X$$

The expectation value X or more generally of $g(X)$ becomes:

$$\mathrm{E}\big(g(X)\big) = \int_{-\infty}^{+\infty} g(X)dF$$

When X range over a continuous range and $F(x)$ is continuous with a derivative $f = dF/dx =$ the **probability density function** (PDF) we get

$$\mathrm{P}(a < x \leq b) = \int_a^b f dx \quad \text{and} \quad \mathrm{E}\big(g(X)\big) = \int_{-\infty}^{+\infty} g \cdot f dx$$

where the density function $f = F'$ acts like a weight function.

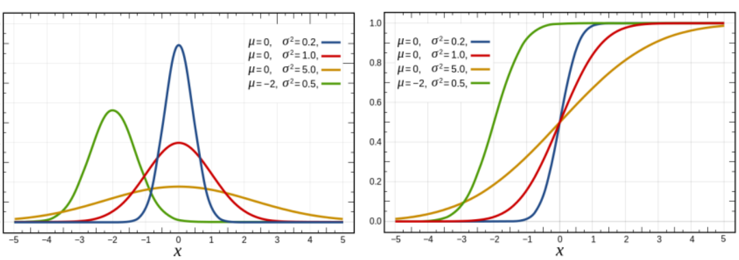


Fig. 3.7.6 Probability densities $f = F'$ and their cumulative distributions F.

$$\alpha'(x) = \frac{d\alpha}{dx} \text{suggests} \int_a^b f d\alpha = \int_a^b f\alpha' dx$$

This is true if $f \in \mathrm{C}^0$ and $\alpha \in \mathrm{C}^1$. A weaker demand on $\alpha(x)$ that includes jumps and guarantees existence of $\int_a^b f d\alpha$ for every $f \in \mathrm{C}^0$ is that $\alpha(x)$ is of bounded variation. Which means that the vertical distance covered on the graph when going from $\big(a, \alpha(a)\big)$ to $(b, \alpha(b))$ is finite.

Devil's staircase

The devil's staircase also known as the Cantor function $c(x)$ is a continuous and increasing function from [0,1] to [0,1] which makes it a natural CDF of a random variable X with values in [0,1]. What makes it interesting is that it has zero derivative almost everywhere. The PDF of X, $c'(x)$ is zero at every point where it is defined which is a subset of [0,1] of total length one.

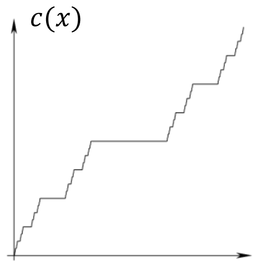


Fig. 3.7.7 Devil's staircase

To define $c(x)$, let $x = (0.x_1x_2x_3\ldots)_3$, $x_i \in \{0,1,2\}$
1. Truncate x by replacing all digits after the first 1 by zeros.
2. Replace all $x_i = 2$ with $\tilde{x}_i = 1$.
3. Reinterpret the digit sequence as a base 2 representation.
$x \curvearrowright 0.x_1 \ldots x_n 100 \ldots (x_i \in \{0,2\}) \curvearrowright 0.\tilde{x}_1\tilde{x}_2 \ldots (\tilde{x}_i \in \{0,1\}) \curvearrowright c(x)$

The definition makes $c(x)$ flat in the middle third and the definition is self-similar for the surrounding thirds, just a move forward one step in the digit sequence just as in the definition of the Cantor set. $c(x)$ is constant outside the Cantor set. The arc length of the devil's staircase is 2 as it would be for the graph of any staircase from (0,0) to (1,1) but there are no steps or rather an uncountable number of steps of height zero.

The cantor function is uniformly continuous since its domain is a compact set but it fails to be absolutely continuous, an even stronger smoothness property defined by:

For any finite set of disjoint intervals (x_k, y_k) in the domain of $f(x)$:

$$\forall \varepsilon \exists \delta: \sum_k (y_k - x_k) < \delta \Rightarrow \sum_k |f(y_k) - f(x_k)| < \varepsilon$$

Absolute continuity is important in generalizing the fundamental theorem of calculus that connects derivation with Riemann-integration. With Lebesgue integration on the not absolutely continuous Cantor function:

$$1 = \int_0^1 1dc(x) \neq \int_0^1 c'(x)dx = 0$$

A contiuous and non-constant function on $[a, b]$ with a derivative equal to zero outside a set of measure zero is called a singular function.

A function $f(x)$ has bounded variation if $f(x) = g(x) + h(x)$ where both $g(x)$ and $h(x)$ are monotone, (increasing or decreasing).

Another example of RS-integrals comes from physics and finding the balancing point on a rod $[0, L]$ along the x-axis with mass $m(x)$ for the section $[0, x]$, $m(x) = 0$ for $x < 0$ and $m(x) = M = \int_0^L m(x)dx$ for $x > L$. The torque or moment around $\overline{x}$ for a point mass m_j at x_j is proportional to $m_j(\overline{x} - x_j)$. If $\overline{x}$ is the balancing point for n masses $\sum_{k=1}^{n} m_j(\overline{x} - x_j) = 0$ and with both a continuous and discrete distribution for $m(x)$ we get:

$$\overline{x} = \frac{\int_0^L x dm}{\int_0^L dm} = \frac{\int x m' dx}{M} \quad (m'(x) \text{ is linear density for a continuous distribution})$$

In $\mathbb{R}^n$ this translates to a mass-center where all gravity torques are balanced. With local density $\rho(\boldsymbol{r})$ given by I_Ω, the indicator function for a bounded set $\Omega \subset \mathbb{R}^n$ we get the geometric center of an object Ω.

$$\boldsymbol{r}_{MC} = \frac{\int \boldsymbol{r}\rho(\boldsymbol{r})d\boldsymbol{r}}{M} \qquad \boldsymbol{r}_{GC} = \frac{\int \boldsymbol{r} I_\Omega d\boldsymbol{r}}{\int I_\Omega d\boldsymbol{r}} = \frac{\int_\Omega \boldsymbol{r} d\boldsymbol{r}}{\mathrm{Vol}(\Omega)}$$

Theorem. (Riesz' theorem)
If $L: C^0[a, b] \to \mathbb{R}$ is a positive linear functional.
Functional: a function from a vector space (function space) to its scalars.
Linear: $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$
Positive: $\forall x \in [a, b]: f(x) \geq 0 \;\Rightarrow\; L(f) \geq 0$
Then there is an increasing function $\alpha: [a, b] \to \mathbb{R}$ s.t. $L(f) = \int_a^b f(x)d\alpha(x)$.

A simpler version of this states that if L is a positive linear functional with

$$L(f) = (d - c) \text{ for all } f(x) = \begin{cases} 1 \text{ if } x \in [c, d] \subseteq [a, b] \\ 0 \text{ if } x \notin [c, d] \subseteq [a, b] \end{cases} \text{ then } L(f) = \int_a^b f dx$$

This can be applied to work. If we require that $W = Fd$ when a constant force F is applied for a distance d and that work depends linearly on the force then the work of a varying force between a and b must be $W = \int_a^b F dx$.

The Cauchy-Schwarz inequality is one of the most important inequalities in mathematics and a good example of how to use the RS-integral to cover both continuous and discrete versions.

$$u_i, v_i \in \mathbb{R}: \; \left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \sum_{i=1}^{n}(u_i)^2 \sum_{i=1}^{n}(v_i)^2 \qquad \text{Equality iff } \exists(\lambda, \mu) \neq (0,0) \;\; \lambda u_i + \mu v_i = 0$$

$$f, g \in C^0[a, b]: \left(\int_a^b fg\, dx\right)^2 \leq \int_a^b f^2 dx \int_a^b g^2 dx \qquad \lambda f(x) + \mu g(x) = 0$$

Both cases of Cauchy-Riemanns inequality are covered in the following where $\alpha(x)$ is increasing and $f, g \in C^0[a,b]$

$$\underbrace{\int_a^b \left(f(x)\right)^2 d\alpha(x)}_{A} \cdot \underbrace{\int_a^b \left(g(x)\right)^2 d\alpha(x)}_{B} \geq \underbrace{\left(\int_a^b f(x)g(x)d\alpha(x)\right)^2}_{C}$$

Proof.

$\forall\lambda \in \mathbb{R}: (\lambda f + g)^2 \geq 0 \Rightarrow \int_a^b (\lambda f + g)^2 d\alpha \geq 0 \Rightarrow \lambda^2 A + 2\lambda C + B \geq 0$ $A > 0 \Rightarrow \forall\lambda: (A\lambda + C)^2 + AB - C^2 \geq 0 \Rightarrow AB \geq C^2$ (choose $\lambda = -C/A$)
$A = 0 \Rightarrow \forall\lambda: 2\lambda C + B \geq 0 \Rightarrow C = 0 \Rightarrow AB \geq C^2$
Equality iff $\lambda f + g \equiv 0$ ■

3.8 Exponentiation and Logarithms

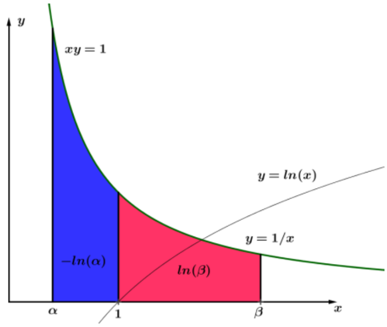
To handle x^y for arbitrary real numbers we need to know how to define and calculate a number like π^π but so far we have not even given a proper definition of muliplication of real numbers. To do this we start with the rational numbers where the operators $+, -, \times, \div$ are easily defined. One way to define $\mathbb{R}$ is as equivalence classes of Cauchy sequences $x = [(x_n)]$ where $x_n \in \mathbb{Q}$ and $|x_m - x_n| \to 0$ as $\min(m,n) \to \infty$. A number can be represented with a sequence from its decimal representation $\pi = [(3, 3.1, 3.14, 3.141, 3.1415, 3.14159 \ldots)]$.

Without going into the details it is now possible to introduce arithmetic on $\mathbb{R}$ by using Cauchy sequences $x + y \equiv [(x_n + y_n)]$, $x \cdot y \equiv [(x_n \cdot y_n)]$ and $x/y \equiv [(x_n/y_n)]$ $(y \neq 0)$ where a representation with $y_n \neq 0$ is chosen. This method runs into problem for x^y since $x_n^{y_n}$ can be irrational, $2^{1/2} \notin \mathbb{Q}$. With arithmetic on $\mathbb{R}$ we can introduce both polynomials $P(x) = \sum_{k=0}^N a_k x^k$ and rational functions $R(x) = P(x)/Q(x)$ defined wherever $Q(x) \neq 0$. x^n is strictly increasing for $n \in \mathbb{Z}^+$and $x \geq 0$ with range $[0, \infty)$. Define $x^{1/n}$ for $n \in Z^+$as the positive y for which $y^n = x$ and then $x^{m/n}$ will be defined for all $m/n \in \mathbb{Q}$ as $\left(x^{1\backslash n}\right)^m$. Any definition of x^y should match this for $y \in \mathbb{Q}$.

The way forward is is to look at the one power function $y = x^k$ for which we lack an antiderivative, $D^{-1}(x^k) = x^{k+1}/(k+1)$ does not work for $k = -1$. We know it exists and equals $f(x) = \int_1^x (1/t)dt + C$ so let us give it a name.

Definition. (Natural logarithm)

$\ln: \mathbb{R}^+ \to \mathbb{R}$ is defined by $\ln(x) \equiv \int_1^x \frac{1}{t}dt$



Theorem.

1. $\ln(x)$ is strictly increasing
2. $\lim_{h\to 0}\frac{\ln(1+h)}{h}=1$
3. $\ln(xy)=\ln(x)+\ln(y)$
4. $f(x)=\ln(x)$ is surjective.

Proof.

1. By definition of $\ln(x)$: $D(\ln(x))=1/x>0\to\ln(x)$ is stricly increasing.

2. $\ln(1)=0$: $\lim_{h\to 0}\frac{\ln(1+h)}{h}=\lim_{h\to 0}\frac{\ln(1+h)-\ln(1)}{h}=\underbrace{\frac{d}{dx}(\ln(x))}_{1/x}\Bigg|_{x=1}=1$

3. $\ln(xy)=\int_1^{xy}\frac{dt}{t}=\int_1^{x}\frac{dt}{t}+\int_x^{xy}\frac{dt}{t}\begin{bmatrix}u=t/x\\dt=xdu\end{bmatrix}=\ln(x)+\int_1^{y}\frac{du}{u}=\ln(x)+\ln(y)$

4. $\ln(x^k)=\ln(x)+\ln(x^{k-1})\to\ln(x^k)=k\cdot\ln(x)$

$$\left.\begin{array}{l}\ln 2>0\to\ln 2^k=k\ln 2\to\infty \text{ as } k\to\infty\\ \ln\frac{1}{2}<0\to\ln\left(\frac{1}{2}\right)^k=-k\ln 2\to-\infty \text{ as } k\to\infty\end{array}\right\}\begin{array}{l}\ln(x) \text{ is continuous}\to\text{its range}\\ \text{is an interval, must be all of }\mathbb{R}.\end{array}$$

Definition. (Euler's number)
The unique real number that satisfies $\ln x=1$ is denoted e.

Euler's number is irrational with numerical value $e=2.718281828\ldots$

Transcendence of e,π and possibly $e+\pi$ and $e\cdot\pi$.

A transcendental number is not the root of a polynomial with rational coefficients. Both e and π were proved to be transcendental in the end of the 19th century. To this date no proof has been given that either $e+\pi$ or $e\cdot\pi$ should be transcendental, it has not even been proved that they are irrational. If a proof appeared it would probably be long and difficult, only understood by experts in the field but a very simple proof can be given that both can't be rational (or algebraic) because if they were then $P(x)=(x-e)(x-\pi)=x^2-(\pi+e)x+\pi\cdot e$ would be polynomial with algebraic coefficients and transcendental roots.

Lindemann's proof that π is not algebraic starts with showing that any number e^x is transcendental when x is algebraic and not zero, $e^{\pi i}=-1$. Alexander Golfand showed that x^y is transcendental if $x\in\mathbb{A}\setminus\{0,1\}$ and $y\in\mathbb{C}\setminus\mathbb{Q}$ which implies that $2^{\sqrt{2}}$ and $e^{\pi}=(-1)^{-i}$ are transcendental.

Definition. (Logarithm and exponential function with base b)
$\log_b : \mathbb{R}^+ \to \mathbb{R}$ is defined by $\log_b(x) \equiv \frac{\ln(x)}{\ln(b)}$ for $b > 0$ and $b \neq 1$
$\exp_b : \mathbb{R} \to \mathbb{R}^+$ is defined as the inverse of $\log_b$.

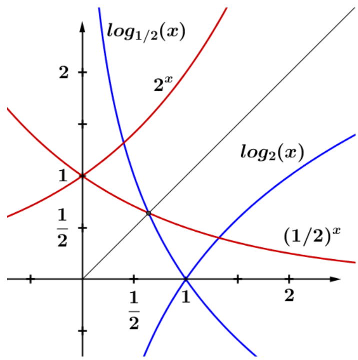
Most frequent are base 2=binary logarithms, base e=natural logarithms and base 10=common logaritms=$\lg(x)$. Since $\log_b$ is continuous so is $\exp_b$.

Theorem.
$\exp_b(x) = b^x$ when $x \in \mathbb{Q}$

Proof.
$\log_b xy = \log_b x + \log_b y \to \log_b x^m = m \log_b x$ for $m \in \mathbb{Z}$
$\log_b y = n \log_b y^{1/n} \to \log_b y^{1/n} = \frac{1}{n} \log_b y$ for $n \in \mathbb{Z}^+$ (by def. of $y^{1/n}$)
$\to \log_b x^{m/n} = \frac{m}{n} \log_b x$ for $m/n \in \mathbb{Q}$
$b^r \curvearrowright r$ by $\log_b \Rightarrow r \curvearrowright b^r$ by $\exp_b$ when $r \in \mathbb{Q}$ ■

Definition.
$b^x \equiv \exp_b(x)$ when $x \in \mathbb{R}$ $(b \in \mathbb{R}^+ \setminus \{1\})$

$\log_b xy = \log_b x + \log_b y$	$b^{x+y} = b^x \cdot b^y$
$\log_b x^y = y \log_b x$	$(b^x)^y = b^{xy}$
$\log_a x = \frac{\log_b x}{\log_b a}$	$a^x = b^{x \cdot \log_b a}$
$\log_b b^x = x$	$b^{\log_b x} = x$



Logarithms are of immense historical importance due to the fact that $\log xy = \log x + \log y$ which sets up a close relation between multiplication and addition; in technical terms, an isomorphism $\log: \mathbb{R}^+ \to \mathbb{R}$ between the groups $(\mathbb{R}^+, \times)$ and $(\mathbb{R}, +)$. To do a multiplication xy before the time of calculators you would look up their logarithms in a table and add them which is much easier than doing a muliplication and then translate back by using the table backwards to get the result: $\exp(\log xy = \log x + \log y) = xy$.

The word *logarithm* has Greek roots, logos meaning proportion and arithmos meaning number. The term stems from John Napier who introduced them in a book from 1614 *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Rule of Logarithms). Natural logarithms are more natural than logarithms of other bases since it comes from the simplest function $1/x$.

The symbol e for the base of the natural logarithm comes from Euler. He used the following definitions for $f(x) = e^x$ and $f^{-1}(x) = \ln x$:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \quad \ln x = \lim_{n\to\infty} n\left(x^{1/n}-1\right)$$

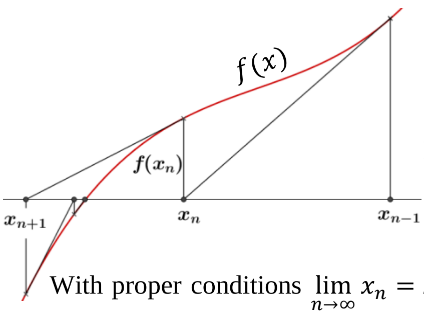
The definition of $\ln x$ with integration gives the following derivatives:

$$D(\ln x) = D\left(\int_1^x \frac{dt}{t}\right) = \frac{1}{x}$$
$$\frac{d(e^x)}{dx} = \frac{1}{d(\ln y)/dy} = \frac{1}{1/y} = y = e^x$$

3.9 Power functions and Roots

After exponential functions $f(x) = b^x$, $b \in \mathbb{R}^+$ and $D_f = \mathbb{R}$ the next step is to investigate power functions $f(x) = x^a \equiv e^{a \ln x}$ with $a \in \mathbb{R}$ and $D_f = \mathbb{R}^+$. This procedure is not practical for calculations since $\ln x$ was defined by a computationally expensive integration and to find e^c with our definition we must solve the equation $\ln x - c = 0$.

Newton's method to find a zero of a function, $f(x) = 0$.



Slope at x_n

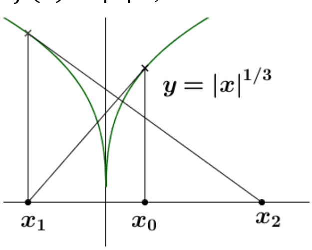
$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

Initiate with a guess x_0 and iterate

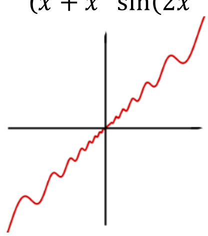
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

With proper conditions $\lim_{n\to\infty} x_n = \hat{x}$ and $f(\hat{x}) = 0$. Some ways it can go wrong is if $f'(x_k) = 0$ for some k, if a cycle occurs x_i=x_j for $i \neq j$, or if the derivative misbehaves around the root. This can lead to divergence of $(x_n)_1^\infty$ no matter how close x_0 is to $\hat{x}$.

$f(x) = |x|^\alpha$, $0 < \alpha <$ ½



$$f(x) = \begin{cases} 0 \text{ if } x = 0 \text{ else} \\ x + x^2 \sin(2x^{-1}) \end{cases}$$



If an iteration starting at x_0 converges to a root $\hat{x}$ then x_0 is said to be in the **basin of attraction,** $\mathcal{B}(\hat{x})$ of that root. Newton's method can be used on complex functions which can lead to very complex basins of attraction in the complex plane.

$f(z) = z^5 - 1 \rightarrow z_{n+1} = \frac{4z_n}{5} + \frac{1}{5{z_n}^4}$

All five basins of attraction have the same boundary, $\partial\mathcal{B}(\hat{z}_1) = \ldots = \partial\mathcal{B}(\hat{z}_5)$.

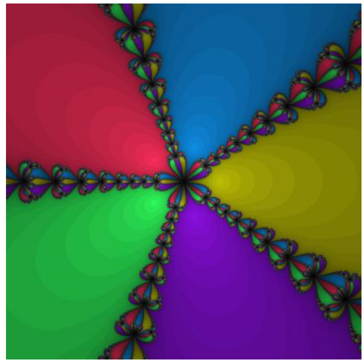


Fig. 3.9.1 Basins of attraction

The multiplicity of a root determines the speed of convergence. A root $\hat{x}$ is of multiplicity k if $D^j f(\hat{x}) = 0$ for $j = 0, \ldots, k-1$ and $D^k f(\hat{x}) \neq 0$. $x^k = 0$ has a root $\hat{x} = 0$ with multiplicity k. The normal case with a root of multiplicity one leads to quadratic convergence which means that the number of correct digits in x_n after a while will tend to double for each iteration. Examples:

a/b can be calculated without using division, $a/b = a \cdot b^{-1}$ and the reciprocal $1/b$ is a root of $f(x) = 1/x - b \rightarrow x_{n+1} = 2x_n - b{x_n}^2$.
$1/3$: $x_0 = 0.5 \rightarrow x_5 = 0.33333333325 \ldots$
With logarithmic tables division can be done with just subtraction and no multiplication, $a/b = e^{(\ln a - \ln b)}$ but a calculation of $\ln x$ is needed.

$$\left.\begin{array}{l} a^b \text{ is a root of } f(x) = x^{1\backslash b} - a \\ \mathrm{D}(x^\alpha) = \mathrm{D}\left(e^{\alpha \ln x}\right) = \cdots = \alpha x^{\alpha-1} \end{array}\right\} \rightarrow x_{n+1} = (1-b)x_n + ab{x_n}^{(1-1/b)}$$

This gives an efficient way of calculating $a^{1/q}$ for $q \in \mathbb{Z}^+$,
$f(x) = x^q - a \rightarrow x_{n+1} = \frac{q-1}{q} x_n + \frac{a}{q} {x_n}^{1-q}$.

With $a = 2$, $q = 2$, $x_0 = 1.5$ and $x_{n+1} = x_n/2 + 1/x_n$

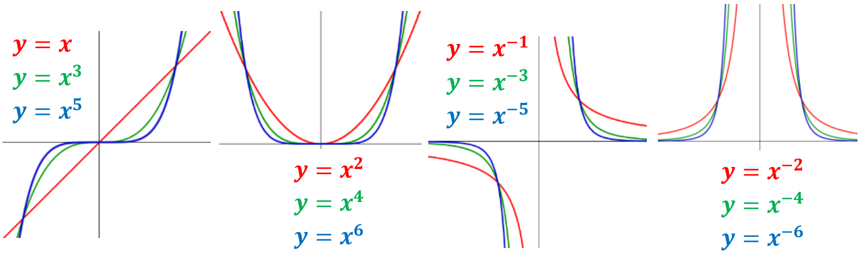
$x_4 = 1.4142135623730950488016896 \ldots$
$2^{1/2} = 1.4142135623730950488016887 \ldots$

Rewriting the exponential laws gives the power laws:
$x^a x^b = x^{a+b} \quad (x^a)^b = x^{ab} \quad (xy)^a = x^a y^a \quad x, y \in \mathbb{R}^+ \;\; a, b \in \mathbb{R}$

With $x^\alpha \equiv e^{\alpha \ln x}$ the power functions $f(x) = x^a$ gets $D_f = \mathbb{R}^+$ from $\ln x$. This definition belongs to analysis with concepts such as real numbers, limits and integration.

If algebra and analysis are looked upon as two parts of the mathematical landscape then they overlap in the definition of exponentiation. We have seen that $x^\alpha \equiv e^{\alpha \ln x}$ and $x^{p/q} \equiv (x^{1/q})^p$ with $y = x^{1/q}$ defined by extraction of positive roots from $y^q = x$ give the same result for positive x. The algebraic definition gives a natural extension to negative x.

$$\alpha \in \mathbb{Z} : f(x) = x^n \to \begin{cases} f(-x) = f(x) \text{ if } n \text{ is even, } f \text{ is an even function.} \\ f(-x) = -f(x) \text{ if } n \text{ is odd, } f \text{ is an odd function.} \end{cases}$$



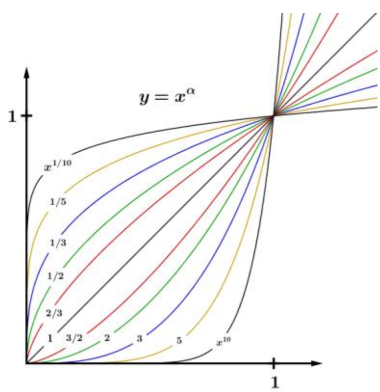
$\alpha \in \mathbb{Q} \setminus \mathbb{Z} \to x^\alpha = (x^{1/n})^m$ with $n \in \{2,3,\ldots\}$ and where $x^{1/n}$ is defined by root extraction from $r^n = x$. Such a root is callen an nth root. To make $x^{1/n}$ a single-valued function, one of the roots must be chosen. It's called the **principal nth root** and it's denoted $\sqrt[n]{x}$. When $n = 2$ it's the square root and the index is usually dropped, for $n = 3$ it's the cube root. The symbol $\sqrt{\ }$ is called radical or radix with radicand as content. Its origin is uncertain but it may stem from the letter r from the latin word *radix* that means root.

For positive x the choice of root is natural $\sqrt[n]{x} = e^{(\ln x)/n} \in \mathbb{R}^+$. For negative x and odd n, the negative root is sometimes seen $\sqrt[n]{-|x|} = -\sqrt[n]{|x|}$ but just as often you see the complex root with least argument in $[0,2\pi)$. With $z = Re^{i\theta}$ and $\theta \in [0,2\pi)$ one gets $\sqrt[n]{z} = R^{1/n}e^{i\theta/n}$. No choice can extend $(xy)^a = x^a y^a$ to negative x, y. $(-1)^{1/2} \cdot (-1)^{1/2} \neq (-1 \cdot (-1))^{1/2}$

Just as rational numbers p/q are easy to distinguish by presenting them in reduced form with $(p,q) = 1$ there is a simplifed form for expressions with radicals having:

- no factor in the radicand of $\sqrt[n]{\cdot}$ with a power $\geq n$.
- no fractions in the radical.
- no radicals in the denominator

$$\frac{\sqrt[3]{16\pi/5}}{\sqrt{3}} = \frac{2}{15} \cdot \sqrt{3} \cdot \sqrt[3]{50\pi}$$



Expressions with sums of radicals in the denominator or nested radicals are much harder to simplify, if at all possible.

$x^3 + px + q = 0 \rightarrow$ One root is $x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$

$$\sqrt{\sqrt[3]{28} - \sqrt[3]{27}} = \left(\sqrt[3]{98} - \sqrt[3]{28} - 1\right)/3$$

An algorithm to decide when a nested radical can be denested was first given in 1989 by Susan Landau, the Landau algorithm.

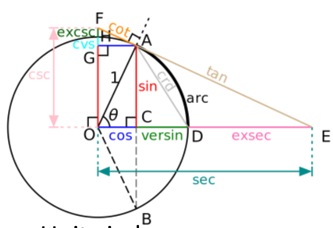
y=e^x and y=ln x are examples of non-algebraic functions or transcendental functions, they don't satisfy a polynomial equation $\sum_{k=0}^{n} a_k(x) y^k = 0$ where $a_k(x) \in \mathbb{Q}[X]$ (polynomials with rational coefficients). Polynomial functions and rational functions with rational coefficients are algebraic functions of degree n=1 since they are solutions to $y - P(x) = 0$ and $Q(x)y - P(x) = 0$. The n-th root of a polynomial is algebraic of degree n, $y^n - P(x) = 0$. Care must be taken for cases with multiple solutions like $y^2 - x^2 = 0$, $y = |x|$ is not algebraic but a combination of two different branches $y = \pm x$.

Every function obtained from a finite sequence of steps using $+, -, \times, \div$ and $\sqrt[n]{\cdot}$ is algebraic but there are also algebraic functions that do not belong to this group. An example is the Bring radical $y(x)$ which satisfies $y^5 + y + x = 0$ and where $y(1)$ is the unique real root of $y^5 + y + 1 = 0$. By Galois theory it can't be solved in closed form with radicals. Other examples of transcendental functions are the trigonometric functions and their inverses.

3.10 Trigonometry

Trigonometry is an ancient branch of mathematics. The word comes from the Greek terms, *trigon* meaning triangle and *metron* meaning measure. The corner stone of Euclidean geometry is the Pythagorean theorem of a right angled triangle that relates the hypothenuse to the catheti, $c^2 = a^2 + b^2$. This was known long before the time of Pythagoras ~500 BC. Even older are the Sumerian roots of the common angle unit with $360° = 2^3 \cdot 3^2 \cdot 5$ degrees for a complete turn. A very practical number with nice integer values for common angles like $10°, 15°, 30°, 45°, 60°$ and $90°$.

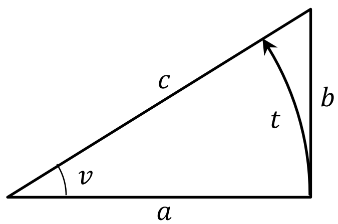
There are a lot of trigonometric function for calculating sides from angles of triangles but all of them can be expressed in terms of one of them. A good choice is to stick to sin, cos, tan and possibly $\sec\theta \equiv \frac{1}{\cos\theta}$ and $\cot\theta \equiv \frac{1}{\tan\theta}$.



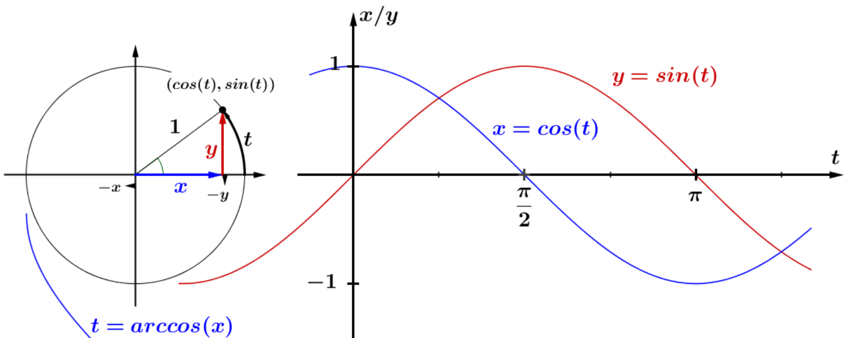
Unit circle

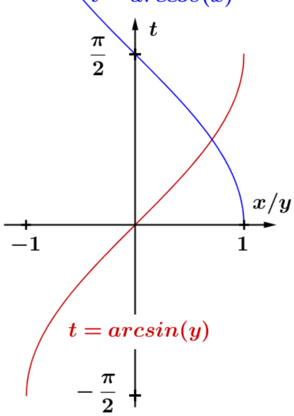
The first trigonometric table was made by Hipparchus' in 140BC and the most famous is Ptolemy's table of chords in *Almagest* from the 2nd century. In those days trigonometry was a tool for astronomy and spherical geometry.

The definition of trigonometric functions can be given in terms of a right-angled triangle or a unit circle to extend the range to all of $\mathbb{R}$. There are many choices for the definition of angles. One full turn could be $360°, 1$ or anything else but there is only choice that is natural for calculus and that is radians, defined as arclength/radius for an angular sector. π is defined as circumference/diameter which is a mistake since it makes $360° = 2\pi$ rad. Every formula of mathematics and physics with an underlying rotational symmetry contains 2π instead of π as it would have been with a more natural definition of pi based on radius instead of diameter.



$$a^2 + b^2 = c^2$$
$$\cos v \equiv a/c$$
$$\sin v \equiv b/c$$
$$\tan v \equiv b/a$$
$$v = t/a \textbf{ rad}$$





Symmetries

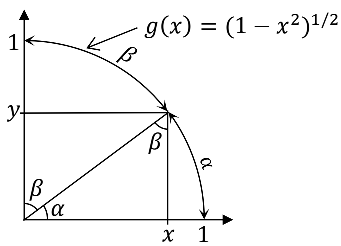
$\cos(t + n \cdot 2\pi) = \cos t$	$\sin(t + n \cdot 2\pi) = \sin t$
$\cos(t + \pi) = -\cos t$	$\sin(t + \pi) = -\sin t$
$\cos(-t) = \cos t$	$\sin(-t) = -\sin t$
$\cos(\pi/2 - t) = \sin t$	$\sin(\pi/2 - t) = \cos t$

$$x^2 + y^2 = 1^2 \rightarrow$$
$$\cos^2 t + \sin^2 t = 1$$

The inverse trigonometric functions $\operatorname{arctrig}(x)$ are obtained after restricting the domain of $\operatorname{trig}(x)$ to a suitable interval where it is strictly monotonic. The prefix arc- of the inverses is suitable since their ranges are over angles that are arc length of the unit circle. As for every inverse function their graph can be obtained by reflection in the diagonal line $y = x$.

Inverse trigonometric functions have an alternative notation which can be confusing since the exponent in $\operatorname{trig}^{-1}(x) = \operatorname{arctrig}(x)$ refers to composition of functions $(f \circ g)(x) = f(g(x))$ whereas the exponent in $\operatorname{trig}^{n}(x)$ refers to multiplication of functions $(f \cdot g)(x) = f(x) \cdot g(x)$.

To verify existence and properties of trigonometric functions we need a good definition and to do that we start by defining $x \curvearrowright \arcsin x$.



$$\beta = \int_0^x \sqrt{1 + \left(g'(u)\right)^2}\, du = \int_0^x \frac{du}{\sqrt{1 - u^2}}$$

$$\arcsin x \equiv \int_0^x \frac{du}{\sqrt{1 - u^2}} \quad \text{for } x \in (-1,1)$$

$$\pi \equiv 2 \cdot \lim_{x \to 1} \int_0^x \frac{du}{\sqrt{1 - u^2}}$$

Extend the definition to $[-1,1]$ with $1 \curvearrowright \pi/2$ and $-1 \curvearrowright -\pi/2$, $\arcsin x$ is a continuous function, $D(\arcsin x) = (1 - x^2)^{-1/2} > 0 \rightarrow$ stricly increasing with domain $[-1,1]$ and range $[-\pi/2, \pi/2]$.

Define $\sin x$ as the inverse of $\arcsin x$ and let $\cos x \equiv \sqrt{1 - \sin^2 x}$. By using the symmetries from the previous page their domain can be extended from $[-\pi/2, \pi/2]$ to all of $\mathbb{R}$ without losing continuity, also their derivatives of all orders will remain continuous.

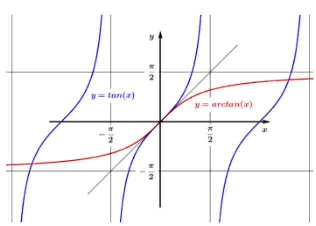
$$y = \sin x \rightarrow \frac{d}{dx}(\sin x) = \frac{1}{\frac{d}{dy}(\arcsin y)} = \sqrt{1 - y^2} = \cos x$$

$$\alpha + \beta = \arccos x + \arcsin x = \pi/2 \rightarrow D(\cos x) = -\sin x$$

$$D(\sin x) = \cos x \qquad D^2(\sin x) = -\sin x \qquad D^3(\sin x) = -\cos x \qquad \cdots$$

$$D(\cos x) = -\sin x \qquad D^2(\cos x) = -\cos x \qquad D^3(\cos x) = \sin x \qquad \cdots$$

$\tan x \equiv \sin x \,/\, \cos x$ with domain
$\{x \in \mathbb{R} | x \neq \pm\pi/2 + n \cdot 2\pi\}$
$D(\tan x) = 1 + \tan^2 x$
$D(\arctan x) = 1/(1 + x^2)$



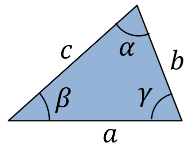
There are many trigonometric identities that are helpful for problem solving.

Law of sines:

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}=\frac{2A}{abc}$$

A =Area of triangle

$s=(a+b+c)/2$



Area:

$$A=\frac{ab\sin\gamma}{2}=\frac{bc\sin\alpha}{2}=\frac{ca\sin\beta}{2}=\sqrt{s(s-a)(s-b)(s-c)}$$

Law of cosine (An extension of Pythagoras theorem):

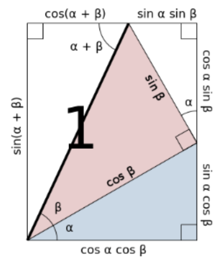
$$c^2=a^2+b^2-2ab\cos\gamma$$

Trigonometric addition formulas:

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta)=\cos\alpha\, cos\beta\mp\sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$



These formulas are simply another way of saying that a rotation of α degrees followed by a rotation of β degrees equals one rotation of $\alpha+\beta$ degrees.

$$\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\vec{v}=\begin{pmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{pmatrix}\vec{v}$$

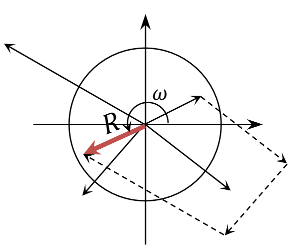
Another way of seeing this is to study $e^z e^w=e^{z+w}$ for $z,w\in\mathbb{C}$ and look at the real and imaginary parts of $e^{i(\alpha+\beta)}=e^{i\alpha}e^{i\beta}$. This also gives de Moivres formula: $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$.

$$\sin(n\theta)=\sum_{\substack{k\text{ odd}\\ k\le n}}(-1)^{\frac{k-1}{2}}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta\rightarrow\sin(2\theta)=2\sin\theta\cos\theta$$

$$\cos(n\theta)=\sum_{\substack{k\text{ even}\\ k\le n}}(-1)^{k/2}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta\rightarrow\cos(2\theta)=\cos^2\theta-\sin^2\theta$$

$$\cos 2\theta=\begin{cases}2\cos^2\theta-1\rightarrow\cos(\theta/2)=(\pm)\sqrt{(1+\cos\theta)/2}\\ 1-2\sin^2\theta\rightarrow\sin(\theta/2)=(\pm)\sqrt{(1-\cos\theta)/2}\end{cases}$$

(Sign based on θ)



$$\sum_{i=1}^{n}a_i\cos(x+\varphi_i)+\sum_{k=1}^{m}\underbrace{b_k\sin(x+\theta_k)}_{\hat{b}_k\cos(x+\hat{\theta}_k)}=R\cos(x+\omega)$$

Power reducing formula (examplified with for $\cos^n \theta$ and n odd).

$$2^n \cos^n \theta = 2 \sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} \cos(\,(n-2k)\theta)$$

Product-to-Sum

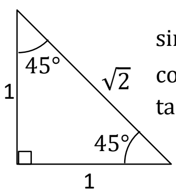
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \cos(\alpha-\beta)$$

Sum-to-Product

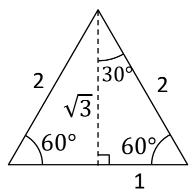
$$\cos\alpha + \cos\beta = 2\cos\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right)$$
$$\sin\alpha + \sin\beta = 2\sin\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right)$$
$$\sin\alpha - \sin\beta = 2\sin\left(\tfrac{\alpha-\beta}{2}\right)\cos\left(\tfrac{\alpha+\beta}{2}\right)$$
$$\cos\alpha - \cos\beta = 2\sin\left(\tfrac{\alpha+\beta}{2}\right)\sin\left(\tfrac{\beta-\alpha}{2}\right)$$

$$2^n \prod_{k=1}^{n} \cos\alpha_k = \sum_{\vec{e}\,\in\,\{-1,1\}^n} \cos\left(\sum_{k=1}^{n} e_k \alpha_k\right)$$

The values of trigonometric functions for some arguments should be known, others can be calculated when needed with the trigonometric identities.



$\sin 45° = 1/\sqrt{2}$
$\cos 45° = 1/\sqrt{2}$
$\tan 45° = 1$



$\sin 30° = 1/2$ $\quad \sin 60° = \sqrt{3}/2$
$\cos 30° = \sqrt{3}/2$ $\quad \cos 60° = 1/2$
$\tan 30° = 1/\sqrt{3}$ $\quad \tan 60° = \sqrt{3}$

Rational fractions of a lap, $2\pi p/q$ are mapped by cos and sin to algebraic numbers, this follows from de Moivre's formula, $z = e^{i\cdot 2\pi p/q} \to z^q - 1 = 0$. Some of the oldest problems of mathematics are tied to the form of these values. Greek geometry was based on constructions with ruler and compass. The ruler corresponds to linear constructions and linear eauations while the compass corresponds to, $(x-x_0)^2 + (y-y_0)^2 = r_0{}^2$. Any point (x, y) that can be constructed with these tools in a finite number of steps must be based on $+, -, \times, \div$ and $\sqrt{\cdot}$. Examples of such problems are:

- Drawing a square with the same area as a circle ($r = 1 \to s = \sqrt{\pi}$).
- Drawing a cube with twice the volume of a given cube ($s = 1 \to S = 2^{1/3}$).
- For which n can you construct a regular n-gon. Is the 7-gon constructible?
- Divide a given angle into three equal angles. Can $60°$ be trisected?

For 2000 years constructions were sought. Now we know that they can not exist. $\sqrt{\pi}$ and $2^{1/3}$ do not belong to the minimal field $F \supseteq \{a + bi | a, b \in \mathbb{Q}\}$ that is closed under taking square roots, $x \in F \Rightarrow \sqrt{x} \in F$.

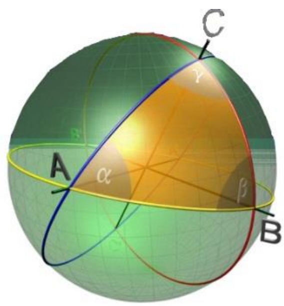
Most angles can't be trisected, it depends on whether $\cos(\theta/3)$ is expressible with arithmetics and square roots on $\mathbb{Q}$. $\cos 3\theta = 4\cos^3(\theta) - 3\cos(\theta)$ makes $x = \cos 20°$ a solution to $4x^3 - 3x - 1/2 = 0$. This equation has no rational roots that can be factored out to make $\cos 20°$ constructible with square roots. There can be no general procedure for trisecting an angle.

Gauss proved that a regular 17-gon can be constructed by showing that:

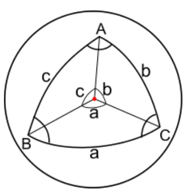
$$16\cos\left(\frac{2\pi}{17}\right) = -1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}$$

He also proved which regular n-gons can be constructed. For $\cos(2\pi/n)$ to be constructible, n must be of the form $2^k p_1 p_2 \dots p_l$ with $k, l \in \{0,1,\dots\}$ and each p_j being a distinct Fermat prime. A Fermat prime is a prime number of the form $F_m = 2^{2^m} + 1$. Fermat conjectured that every Fermat number was a prime number $F_m \in \{3, 5, 17, 257, 65537, 4294967297, \dots\}$. Seventy years after Fermat's death Euler proved him wrong $F_5 = 641 \cdot 6700417$ and so far there has not been found any more Fermat primes beyond F_4. Construction of a regular F_4-gon was given by J. Hermes in 1894, it took him 10 years and 200 pages.

Trigonometry is not only useful for Euclidean geometry. Much of early trigonometry was used used for astronomy and astrology with planets and stars on the celestial sphere. Straight lines in the plane corresponds to great circles and triangles are made from segments of such circles. With journeys over the oceans came the need for navigation that required calculations with spherical trigonometry.



Spherical geometry does not follow Euclid's fifth postulate, a version of which says that any line has one parallel line going through an outside point. Geometry on a sphere belongs to elliptic geometry where thera are no parallel lines. As a consequence the angle sum of a triangle is not 180° but always $\alpha + \beta + \gamma > 180°$. Geometry is no longer independent of scale. If a triangle is scaled down $\alpha + \beta + \gamma \to 180°$ as the area goes to zero. Euclidean geometry is regained in the limit approaching the tangential plane.



Unit sphere

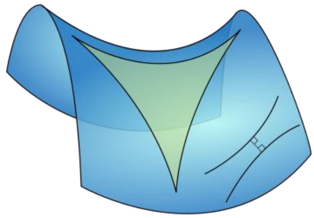
Spherical law of cosines:

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

+ cyclic permutations of a,b,c and A,B,C

Spherical law of sines:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$



In geometry on a hyperbolic surface of constant negative curvature there are more than one non-intersecting line to a given line through a point outside the line. A sphere of radius R has constant curvature $K = 1/R^2$. A pseudosphere of radius R is a surface with a saddle shape and constant curvature $K = -1/R^2$.

The line segments of a hyperbolic triangle are formed by geodetic curves, the shortest curve connectiong two points. The segment is unique and can be extended indefinitely in both directions.

There are many similarities between spherical and hyperbolic geometry. For spheric/hyperbolic triangles and circles with angles α, β, γ and radii r, drawn on spheres/psuedosheres of radius R:

	Spherical geometry	**Hyperbolic geometry**
Angle sum:	$\alpha + \beta + \gamma > \pi$	$\alpha + \beta + \gamma < \pi$
Triangle area:	$R^2(\alpha + \beta + \gamma - \pi)$	$R^2(\pi - \alpha - \beta - \gamma)$
Circle circumference:	$2\pi R \sin(r/R)$	$2\pi R \sinh(r/R)$

Trigonometric functions in ordinary geometry are based on the unit circle $x^2 + y^2 = 1$. In hyperbolic geometry there is another kind of trigonometric functions called hyperbolic functions. They are based on the unit hyperbola $x^2 - y^2 = 1$ with $\cosh^2 t - \sinh^2 t = 1$. The functions hyperbolic cosine and hyperbolic sine are pronounced "kosh" and "sinch".

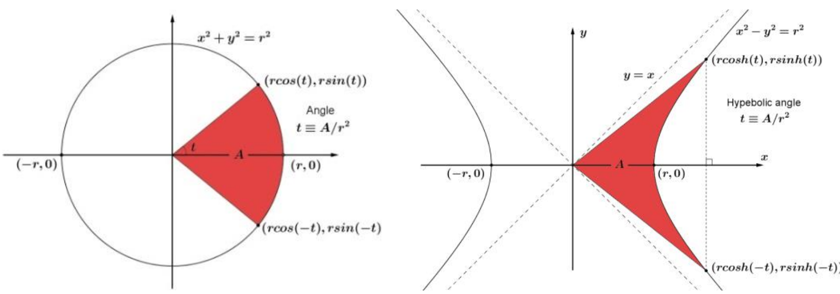
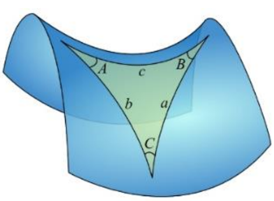


Fig. 3.10.1 Definition of trigonometric functions, ordinary and hyperbolic.



Hyperbolic laws of cosines and sines:

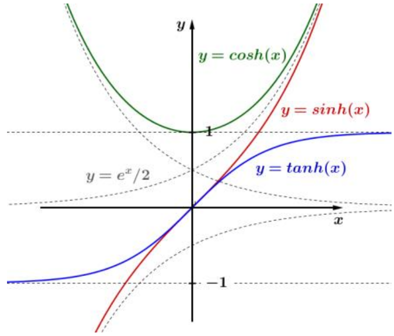
$\cosh\frac{a}{R} = \cosh\frac{b}{R}\cosh\frac{c}{R} - \sinh\frac{b}{R}\sinh\frac{c}{R}\cos A$

$\cos A = -\cos B \cos C + \sin B \sin C \cosh\frac{a}{R}$

$\sin A/\sinh\frac{a}{R} = \sin B/\sinh\frac{b}{R} = \sin C/\sinh\frac{c}{R}$

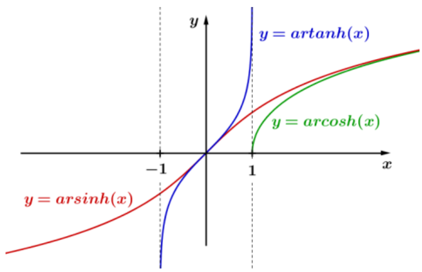
Hyperbolic functions have properties resembling the trigonometric functions:

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$
$$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$
$$\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$$
$$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$$
$$\mathrm{D}(\sinh x) = \cosh x$$
$$\mathrm{D}(\cosh x) = \sinh x$$
$$\mathrm{D}(\tanh x) = 1 - \tanh^2 x$$



The inverse trigonometric functions had the prefix arc- since their argument was an angle defined by an arc. The inverse hyperbolic functions have the prefix ar- since their angle is defined by an area.

$$\mathrm{D}\big(\text{arsinh}\,(x)\big) = 1/\sqrt{x^2+1}$$
$$\mathrm{D}\big(\text{arcosh}\,(x)\big) = 1/\sqrt{x^2-1}$$
$$\mathrm{D}(\text{artanh}\,(x)) = 1/(1-x^2)$$



3.11 Power series

There must be a reason behind all the similarities between trigonometric laws and hyperbolic laws. The hyperbolic laws are easy to prove once you have done exercise 3.34 and showed that:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

Another way to say this is that $\cosh x$ equals the symmetric part of e^x and $\sinh x$ equals the anti-symmetric part of e^x. The property of integers and functions to be even or odd is called **parity**. Every function can be written as a sum of an even a.k.a. symmetric function and an odd a.k.a. anti-symmetric function, $f = f_S + f_A$. This decomposition is unique since $f_S + f_A = g_S + g_A$ implies $f_S - g_S = g_A - f_A$ which means $f_S - g_S$ and $g_A - f_A$ must be both even and odd, $h(x) = h(-x) = -h(-x) \Rightarrow h(x) = 0 \rightarrow g_S = f_S \wedge g_A = f_A$.

The trigonometric functions are defined from the circle $x^2 + y^2 = 1$ whereas the hyperbolic functions are defined from the hyperbola $x^2 - y^2 = 1$. One of these can be obtained from the other by a replacement $y \curvearrowright iy$. Could the similar laws be explained by extending the domains of $\cos x$, $\sin x$ and e^x to complex numbers, as we can do for polynomials $P(z) = \sum_{k=0}^{n} a_k z^k$?

The way to connect general functions to polynomials is via approximations. Assume $f(x)$ is n times differentiable in an open interval containing $x = a$ and you want to imitate $f(x)$ in a neighborhood of a with polynomials of ever higher degree $P_m(x) = \sum_{k=0}^{m} a_k (x-a)^k$ to match the shape of $f(x)$.

m
$0: P_0(a) = f(a) \Rightarrow \qquad P_0(x) = f(a)$
$1: P_1(a) = f(a), {P_1}'(a) = f'(a) \Rightarrow \quad P_1(x) = f(a) + f'(a)(x-a)$
$2: {P_2}^{(k)}(a) = f^{(k)}(a), k = 0,1,2 \Rightarrow \quad P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$
… …
$$n: {P_n}^{(k)}(a) = f^{(k)}(a), k = 0,1,\dots,n \Rightarrow \qquad P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

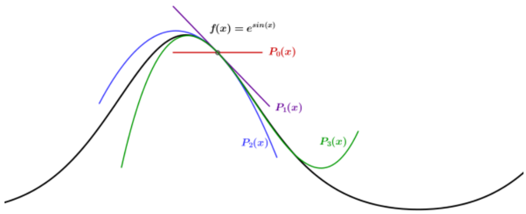


Fig. 3.11.1 $e^{\sin x}$ with Taylor polynomials of degree 0,1,2 and 3 around $x = 2$.

Theorem. (Taylor's formula)
If $f \in C^n(I, \mathbb{R})$ for some open interval I containing a and x then
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$$
$R_n(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n$ for some $\xi \in [a,x] \cup [x,a]$
$R_n(x)$ is called Lagrange's remainder term.

Another form of $R_n(x)$ is $(x-a)^n A(x)$ with A bounded for closed intervals. Formulae with $a = 0$ are named after Maclaurin. Taylor (1685-1731) was from England and Maclaurin (1698-1741) was from Scotland.

$$f(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(\xi)}{n!}x^n, \xi \in [0,x]$$

Proof.
Repeated use of integration by parts $\left(\int_a^b f \cdot g dx = [F \cdot g]_a^b - \int_a^b F \cdot g' dx\right)$
$$f(x) - f(a) = \int_a^x 1 \cdot f'(t)dt = [(t-x)f'(t)]_a^x - \int_a^x (t-x)f''(t)dt =$$

$$f(x)-f(a)=(x-a)f'(a)-\left[\frac{(t-x)^2}{2}f''(t)\right]_a^x+\int_a^x\frac{(t-x)^2}{2}f'''(t)dt$$

$$=(x-a)f'(a)+\frac{(x-a)^2}{2}f''(a)+\cdots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n(x)$$

$$R_n(x)=\int_a^x\frac{(x-t)^{n-1}}{(n-1)!}f^{(n)}(t)dt$$

> The mean value theorem of integral calculus:
> If $f\in \mathrm{C}^0([a,b],\mathbb{R})$and g is integrable with no change of sign in [a, b]then
> $\int_a^b f(x)g(x)dx=f(c)\int_a^b g(x)dx$ for some $c\in[a,b]$

$$R_n(x)=f^{(n)}(\xi)\int_a^x\frac{(x-t)^{n-1}}{(n-1)!}dt=\frac{f^{(n)}(\xi)}{n!}(x-a)^n \text{ for some } \xi\in[a,x] \ \blacksquare$$

$R_n(x)=f^{(n)}(\xi)(x-a)^n/n!$ and $f^{(n)}$ continuous and bounded in any closed interval has given rise to a shorter notation, the big-O notation that describes the limiting behavior of a function.

If $\exists M,x_0$ s.t. $|f(x)|\le M|g(x)|$ for $x>x_0$: $f(x)=O(g(x))$ as $x\to\infty$
If $\exists M,\delta$ s.t. $|f(x)|\le M|g(x)|$ for $0<|x-x_0|<\delta$: $f(x)=O(g(x))$ as $x\to a$

Apart from truncated Taylor series and asymptotic limits, the big-O notation is often used to describe how the number of steps or memory usage in algorithms increase with input size. Exponential growth $O(a^n)$ is always faster than polynomial growth $O(n^b)$. Any function growing faster than n^c for any c is superplynomial and any function growing slower than c^n for any c is subexponential.

If there is a big-O notation theree should be a little-O notation. There is, same definition but with "$\exists M>0$" replaced by "$\forall\varepsilon>0$". $f(x)=o(g(x))$ means $f(x)$ grows much slower than $g(x)$, $\sum_{k=0}^n a_kx^k=o(x^{n+\varepsilon})$ for any $\varepsilon>0$.

$\mathrm{D}(e^x)=e^x$ makes it easy to get the Taylor series of e^x around $x=0$.

$$e^x=\sum_{k=0}^{n-1}\frac{x^k}{k!}+R_n(x) \text{ with } R_n(x)=\frac{e^\xi}{n!}x^n \text{ for some } \xi\in[0,x]$$

$$\lim_{n\to\infty}|R_n(x)|\le e^{|x|}\lim_{n\to\infty}\frac{|x|^n}{n!}\le e^{|x|}\lim_{n\to\infty}\frac{|x|^n}{(n/2)^{n/2}}=e^{|x|}\lim_{n\to\infty}\left(\frac{x^2}{n/2}\right)^{n/2}=0$$

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

Derivatives and antiderivatives			
$f(x)$	$\mathrm{D}(f(x))$	$\mathrm{D}^n(f(x))$	$D^{-1}(f(x))$
x^α	$\alpha x^{\alpha-1}$	$\alpha^{\underline{n}} x^{\alpha-n}$	$\alpha^{-1} x^{\alpha+1}$
$\ln x$	$1/x$	$(-1)^{n-1}(n-1)!/x^n$	$x\ln x - x$
e^x	e^x	e^x	e^x
$\sin x$	$\cos x$	$+/-\quad \cos x \ / \ \sin x$	$-\cos x$
$\cos x$	$-\sin x$	$+/-\quad \cos x \ / \ \sin x$	$\sin x$
$\tan x$	$1/\cos^2 x$	$P_{n+1}(\tan x)$	$-\ln\lvert\cos x\rvert$
$\arcsin x$	$(1-x^2)^{-1/2}$	…	$x\arcsin x + \sqrt{1-x^2}$
$\arccos x$	$-(1-x^2)^{-1/2}$	…	$x\arccos x - \sqrt{1-x^2}$
$\arctan x$	$(1+x^2)^{-1}$	…	$x\arctan x - \ln\sqrt{x^2+1}$
$\sinh x$	$\cosh x$	$\sinh x \ / \ \cosh x$	$\cosh x$
$\cosh x$	$\sinh x$	$\cosh x \ / \ \sinh x$	$\sinh x$
$\tanh x$	$1/\cosh^2 x$	$P_{n+1}(\tanh x)$	$\ln\cosh x$
$\operatorname{arsinh} x$	$(x^2+1)^{-1/2}$	…	$x \operatorname{arsinh} x - \sqrt{x^2+1}$
$\operatorname{arcosh} x$	$(x^2-1)^{-1/2}$	…	$x \operatorname{arcosh} x - \sqrt{x^2-1}$
$\operatorname{artanh} x$	$(1-x^2)^{-1}$	…	$x \operatorname{artanh} x + \ln\sqrt{1-x^2}$

Power series expansions $f(x) = \sum_{k=0}^{n-1} f^{(k)}(0)x^k/k! + R_n(x)$	Interval where $\lim_{n\to\infty} R_n(x) = 0$
$(x+1)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + R_{n+1}(x)$	$\alpha>0: x \in [-1,1]$ $\alpha<0: x \in (-1,1)$
$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + R_{n+1}(x)$	$x \in (-1,1]$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x)$	$\lvert x\rvert < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + R_{2n+3}(x)$	$\lvert x\rvert < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + R_{2n+2}(x)$	$\lvert x\rvert < \infty$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + R_{2n+3}(x)$	$\lvert x\rvert < \infty$
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + R_{2n+2}(x)$	$\lvert x\rvert < \infty$

n.b. $\binom{\alpha}{k} \equiv \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)}{k!}$ $(\alpha \in \mathbb{R})$ and eq. 1 $\to \frac{1}{1-x} = 1 + x + x^2 + \cdots$

Example. (e is not rational)

$$x = 1 \to e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + R_n(1)$$

$$r = R_n(1) = e^{\xi}/n! \text{ and } 0 \le \xi \le 1 \to 0 < r < 3/n!$$

Assume $e = p/q$ and let $n - 1 \ge q$. Multiply both sides with $(n-1)!$.
$r(n-1)!$ must be integer but $0 < r(n-1)! < 3/n$ for all $n \Rightarrow e \notin \mathbb{Q}$. ■

Taylor's theorem is very useful for studying extrema and limits of functions. When an expression is built from several functions it is often a good idea to replace them with truncated Taylor series' and big O-notations for remaining terms. Complicated expressions with several O-terms can be simplified more radically than expressions with ordinary terms as long as the rules of O-notation are followed. An expression containing an O-term represents a set of functions with certain limiting behavior near a point or at infinity. An equation between two such expressions $A = B$ really means $A \subseteq B$ and if you can prove something for functions in B this will apply to all functions in A as well. For instance $(n + O(n^{1/2}))(n + O(\log n))^2 = n^3 + O(n^{5/2})$ as $n \to \infty$.

From Taylor's formula and the remainder term $f^n(\xi)(x - x_0)^n/n!$ follows:

Theorem.
If $f \in C^n(I, \mathbb{R})$ in a neighborhood I of x_0 and
$f'(x_0) = f''(x_0) = \cdots = f^{n-1}(x_0) = 0$ while $f^n(x_0) \ne 0$ Then

n even and $f^n(x_0) < 0 \Rightarrow f$ has a local strict maximum in x_0.
n even and $f^n(x_0) > 0 \Rightarrow f$ has a local strict minimum in x_0.
n odd $\Rightarrow f$ has neither a local maximum or minimum in x_0.

Even functions have no x^{2k+1}-terms in their Taylor expansions around x_0=0 and odd functions have no x^{2k}-terms.

A smooth function $f \in C^\infty(D)$ on an open set with $x_0 \in D \subseteq \mathbb{R}$ is equal to its **Taylor series** $f(x) = \sum_{k=0}^{\infty} f^k(a)(x - x_0)^k/k!$ whenever $\lim_{n\to\infty} R_n(x)$=0. Any function that can be represented by a convergent power series for every x in a neighborhood of every $x_0 \in D$ is said to be **analytic** in D, $f \in C^\omega(D)$. The coefficients a_k of a function analytic in x_0 (i.e. in a neighborhood of x_0) $f = \sum_{k=0}^{\infty} a_k(x - x_0)^k$ are uniquely determined by f, $a_k = f^k(x_0)/k!$.

$$f \in C^\infty(D) \nRightarrow f \in C^\omega(D)$$

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Smooth does not imply analytic since $f(x) = [x \neq 0] \cdot e^{-1/x^2} \in C^\infty(\mathbb{R})$ but $f^k(0) = 0$ means for f to be analytic at $x = 0$, $f(x) = \sum_{k=0}^{\infty} 0 \cdot x^k \equiv 0$ in a neighborhood of 0 which is not true for f so $f \notin C^\omega(\mathbb{R})$ and more generally $C^0 \subset C^1 \subset C^2 \subset \cdots \subset C^\infty \subset C^\omega$. The letter ω is used since it is also the first infinite ordinal $1,2,3, \ldots, \omega, \omega + 1, \ldots$. The definition of being analytic can be carried over word for word to the world of complex functions $f: \mathbb{C} \to \mathbb{C}$. Complex numbers is a much more natural arena for convergent power series. It gives rise to a deep and rich area of mathematics called complex analysis. A function $f: \mathbb{C} \to \mathbb{C}$ that is complex differentiable in a neighborhood will automatically be infinitely differentiable and equal to its own Taylor series in that neighborhood.

This is a world that we will only take a small glimpse of here by looking at the exponential and trigonometric functions.

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \cos z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \quad \sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

$$\downarrow$$

$$e^{iz} = \cos z + i \sin z$$

The functions will be well-defined and the identity will be valid whenever the series' are convergent. Convergence of a series $\sum_{k=1}^{\infty} a_k$, with $a_k \in \mathbb{R}$ or $\mathbb{C}$ can be tested by comparing to the geometric series $\sum_{k=0}^{\infty} \alpha^k$ that converges whenever $|\alpha| = |a_{k+1}/a_k| < 1$.

Theorem. (Ratio test)

Let $\sum_{k=0}^{\infty} c_k$ be a complex series and $L = \lim_{n\to\infty} |c_{k+1}/c_k|$:

If $L < 1$ then the series converges absolutely $\sum_{k=0}^{\infty} |c_k|$ has a limit.

If $L > 1$ then the series is divergent.

If $L = 1$ or the limit does not exist then the test is inconclusive.

For an absolutely convergent series the sum $S = \sum_{k=0}^{\infty} c_k$ exists and is independent of summation order.

Proof.

If $L = \lim_{k\to\infty} |c_{k+1}/c_k| < 1$ let $r = (L+1)/2$, $L < r < 1$

$\exists N: |c_{n+k}| < r^k |c_n|$ for every $n > N$ and $k > 0 \Rightarrow$

$\sum_{k=N+1}^{\infty} |c_k| = \sum_{k=1}^{\infty} |c_{N+k}| < \sum_{k=1}^{\infty} r^k |c_N| < |c_N| \frac{r}{1-r} < \infty \Rightarrow \sum_{k=1}^{\infty} |c_k| < \infty$

If $L > 1$ then $|c_{k+1}| > |c_k|$ for big enough k will guarantee divergence.

Absolute convergence $\sum_{k=1}^{\infty} |\underbrace{a_k + ib_k}_{c_k}| < \infty \implies$ Convergence $\sum_{k=1}^{\infty} c_k$

$\sum|c_k| < \infty \Rightarrow \sum\sqrt{a_k^2 + b_k^2} < \infty \Rightarrow \sum|a_k| < \infty \wedge \sum|b_k| < \infty$

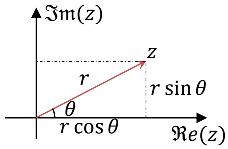
$0 \leq a_k + |a_k| \leq 2|a_k| \rightarrow \hat{S}_n = \sum_{k=1}^{n}(a_k + |a_k|)$ and $\hat{\hat{S}}_n = \sum|a_k|$ are both increasing and bounded → their limit exists → $\sum_{k=1}^{\infty} a_k = \hat{S} - \hat{\hat{S}}$. The same goes for $\sum_{k=1}^{\infty} b_k$. Independence of summation order is left as an exercise.

The ratio test for $e^z, \cos z$ and $\sin z$ gives $L = 0$ and three well-defined functions, analytic in all of $\mathbb{C}$.

$e^z \cdot e^w = e^{z+w}$ (Exercise. 3.15)

$e^{\bar{z}} = \overline{e^z}$

$re^{i\theta} = r(\cos\theta + i\sin\theta)$



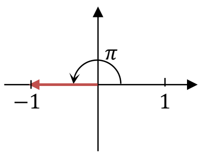
$z = a + bi = re^{i\theta}$

$r = |z| = \sqrt{a^2 + b^2}$

$\theta = \text{atan2}(b, a)$

Function resembling arctan(b/a) that can handle all signs of a, b

From these properties follows all the symmetries and identities of the trigonometric functions.



$e^{\pi i} = -1$

$e^{\pi i} + 1 = 0$

Beautiful formulas connecting analysis (e), geometry (π) and algebra (i) with elementary numbers $(-1,0,1)$.

$e^{z+w} = e^z \cdot e^w \rightarrow$

$\cos(\alpha + \beta) = Re\left(e^{i(\alpha+\beta)}\right) = Re\left(e^{i\alpha} \cdot e^{i\beta}\right) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha + \beta) = Im\left(e^{i(\alpha+\beta)}\right) = Im\left(e^{i\alpha} \cdot e^{i\beta}\right) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$

$e^{nz} = (e^z)^n \rightarrow$

$$\cos n\alpha = Re\left(e^{i\cdot n\alpha}\right) = Re(\cos\alpha + i\sin\alpha)^n = \sum_{\substack{k\in 2\mathbb{N}_0 \\ k\leq n}} \binom{n}{k} (-1)^{\frac{k}{2}} \sin^k\alpha\cos^{n-k}\alpha$$

$$\sin n\alpha = Im\left(e^{i\cdot n\alpha}\right) = Im(\cos\alpha + i\sin\alpha)^n = \sum_{\substack{k\in 2\mathbb{N}_0+1 \\ k\leq n}} \binom{n}{k} (-1)^{\frac{k-1}{2}} \sin^k\alpha\cos^{n-k}\alpha$$

$\cosh z = (e^z + e^{-z})/2 \rightarrow \cosh iz = \cos z \rightarrow \cosh z = \cos iz$

$\sinh z = (e^z - e^{-z})/2 \rightarrow \sinh iz = i\sin z \rightarrow \sinh z = -i\sin iz$

This explains the similarities between hyperbolic and trigonometric laws.

$\cos^2 z + \sin^2 z = (\cos z + i\sin z)(\cos z - i\sin z) = e^z \cdot e^{-z} = e^0 = 1$

$\cosh^2 z - \sinh^2 z = (\cos iz - i\sin iz)(\cos iz + i\sin iz) = 1$

Can we define z^w as $\left(e^{\ln z}\right)^w$? Not really, $f(z) = e^z$ is not injective.

$f(z + 2\pi i) = f(z)$ makes $\ln z$ multivalued $\ln\left(re^{i\theta}\right) = \ln r + i(\theta + n \cdot 2\pi)$.

Big numbers part 3

In the previous part of big numbers from page 91 we saw the Conway chained arrow notation $k_1 \to k_2 \to \cdots \to k_n$. Its recursive definition had an enormous power to generate incredibly huge numbers. In order to generate numbers that are bigger than what we can practically express with horizontal arrows and hierarchical towers of indices counting arrows

$$9 \to^3 9 \equiv 9 \to 9 \to 9 \qquad\qquad 9 \to^{(9\to^{(9\to 9)}9)} 9$$

we will need new and even more powerful ideas.

These new ideas are a combination of:

- Recursion
- A sequence of functions of ever faster growth
- Diagonalization

Diagonalization is the method we saw in Cantors proof from page 101, where real numbers were shown to be uncountable. Any sequence has a first element and every recursion has a base case.

The first step should be as simple as possible and the simplest growing function is the successor function $f_0(n) = n + 1$. The next step in growth after counting is addition which can be defined recursively in terms of successorship. All arithmetic operations can be defined by recursion and the successor function. An axiomatic treatment of this was given by Giuseppe Peano in 1889. It is called Peano arithmetic.

$f_1(m,n) = m + n$	$m + 0 = 0$	$m + f_0(n) = f_0(m + n)$
$f_2(m,n) = m \cdot n$	$m \cdot 0 = 0$	$m \cdot f_0(n) = m + m \cdot n$
$f_3(a,n) = a^n$	$a^0 = 1$	$a^{f_0(n)} = a \cdot a^n$
$f_4(a,n) = a \uparrow\uparrow n$	$a \uparrow\uparrow 0 = 1$	$a \uparrow\uparrow f_0(n) = a^{a\uparrow\uparrow n}$
...	...	...
$f_{p+2}(a,n) = a \uparrow^p n$	$a \uparrow^p 0 = 1$	$a \uparrow^p f_0(n) = a \uparrow^{p-1} (a \uparrow^p n)$

For $k = 3$ and onwards: $f_k(a,0) = 1,\ f_k(a, n+1) = f_{k-1}(a, f_k(a,n))$

A few small modifications give the following sequence of functions:

$$f_0(n) = n + 1$$
$$f_\alpha(n) = f_{\alpha-1}^n(n)$$

$f_0(n) = n + 1$	Succcesor
$f_1(n) = f_0^n(n) = n + n = 2 \cdot n$	Doubling
$f_2(n) = f_1^n(n) = 2^n \cdot n$	$> 2^n$
$f_3(n) = f_2^n(n) = \cdots 2^{2^{2^n n} \cdot 2^n n} \cdot 2^{2^n n} \cdot 2^n n$	$> 2 \uparrow\uparrow n$
...	⋯
$f_\alpha(n) = f_{\alpha-1}^n(n)$	$> 2 \uparrow^{\alpha-1} n$

To move on with a function that grows faster than $f_0, f_1, f_2, f_3, f_4, f_5, \ldots$ requires a new idea, diagonalization.

$f_0(0)$	$f_0(1)$	$f_0(2)$	$f_0(3)$	$f_0(4)$
$f_1(0)$	$f_1(1)$	$f_1(2)$	$f_1(3)$	$f_1(4)$
$f_2(0)$	$f_2(1)$	$f_2(2)$	$f_2(3)$	$f_2(4)$
$f_3(0)$	$f_3(1)$	$f_3(2)$	$f_3(3)$	$f_3(4)$
$f_4(0)$	$f_4(1)$	$f_4(2)$	$f_4(3)$	$f_4(4)$

$$f_\omega(n) \equiv f_n(n)$$

The symbol ω is the same as in C^ω for real analytic functions. It captures an essence of infinity by having a property $P(n)$ for every $n \in \{0,1,2, \ldots\}$. $f_\omega(x)$ will outgrow $f_n(x)$ as soon as $x > n$. Once f_ω is defined the road is open to define $f_{\omega+1}(n) \equiv f_\omega^n(n)$.

Numbers like ω and $\omega + 1$ belong to a class of numbers called ordinal numbers. They start at 0, form an ever increasing sequence and have their own arithmetic: $0,1,2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2, \omega 2 + 1, \ldots$.

Ordinal numbers are closely linked to **well-ordered sets**, sets that have a total order (p. 94) where every non-empty subset has a least element in this ordering. Roughly speaking an ordinal number is an equivalence class of well-ordered sets $(S, <)$ under order preserving bijections:

$$(A, <_A) \cong (B, <_A) \leftrightarrow \exists f: A \to B \text{ s.t } f \text{ bijective}, x <_A y \Rightarrow f(x) <_B f(y)$$

0 represents $\emptyset$ and ω represents $(\mathbb{N}_0, <) \leftrightarrow (0 < 1 < 2 < \cdots)$.
If $\alpha < \beta$ among ordinals then representations can be chosen so that $\alpha \subset \beta$.
$0=\emptyset$ $1=\{0\}$ $2=\{0,1\}$ $3=\{0,1,2\}$ $\omega=\{0,1,2, \ldots\}$ $\omega + 1=\{0,1,2, \ldots, 0'\}$

If the set corresponding to a non-zero ordinal has no maximal element it is a limit ordinal, if not then it is a successor ordinal. Limit ordinals can be written as $\alpha = \lim_n \alpha_n$ of increasing ordinals with $\alpha_n < \alpha$.

Ordinal numbers have their own arithmetic, they can be added, multiplied and exponentiated:

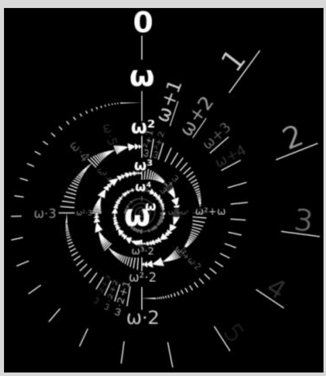
$\alpha_S + \alpha_T$ corresponds to $S \sqcup T$ (disjoint union) with an order operation that keeps old orders and where $(x, y) \in S \times T \Rightarrow x < y$.
$1 + \omega \leftrightarrow \{0', 0,1, \dots\}, 0' < 0$ is isomorphic to $(\mathbb{N}_0, <)$, $\begin{pmatrix} 0' \curvearrowright 0 \\ n \curvearrowright n+1 \end{pmatrix}$ but
$\omega + 1 \leftrightarrow \{0,1,2, \dots, 0'\}, 0' > n$ for $\forall n \in \mathbb{N}_0$ is not isomorphic to $(\mathbb{N}_0, <)$
$1 + \omega = \omega \neq \omega + 1$.

$\alpha_S \cdot \alpha_T$ corresponds to $S \times T$ with lexicographical ordering of pairs (x_i, y_j) with <u>reversed</u> order of significance in the indices.
$\omega 2$:$(0,0) < (1,0) < (2,0) < \cdots < (0,1) < (1,1) < (2,1) < \cdots \leftrightarrow \ \omega + \omega$
2ω:$(0,0) < (1,0) < (0,1) < (1,1) < (0,2) < (1,2) < \cdots \leftrightarrow \omega \neq \omega 2$

${\alpha_S}^{\alpha_T}$ corresponds to the set of functions from T to S with similar ordering as above. For our purposes we can define it inductively $\alpha^0 \equiv 1$ and if the exponent is a successor ordinal $\alpha^{\beta+1} \equiv \alpha^\beta \cdot \alpha$ and if the exponent is a limit ordinal $\alpha^\beta \equiv \lim\limits_{\delta<\beta} \alpha^\delta$.

$\omega + 1, \omega + 2, \dots \dashrightarrow \lim\limits_n (\omega + n) = \omega 2 \dashrightarrow \lim\limits_n \omega n = \omega^2 \dashrightarrow \lim\limits_n \omega^n = \omega^\omega$
$\dashrightarrow \omega^\omega + 1 \dashrightarrow \omega^\omega + \omega \dashrightarrow \omega^\omega \cdot 2 \dashrightarrow \omega^\omega \cdot \omega = \omega^{\omega+1} \dashrightarrow \omega^{(\omega^\omega)} \dashrightarrow$
$\lim\limits_n \omega \uparrow\uparrow n = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} = \varepsilon_0 \dashrightarrow \varepsilon_0 + 1 \dashrightarrow \cdots$ A never ending story.

If α is a limit ordinal $\alpha = \lim\limits_n \alpha_n$ then $f_\alpha(n) \equiv f_{\alpha_n}(n)$ will outgrow f_{α_n} when the input reaches n. To avoid big numbers in our notation we can use α+1 instead of α, $f_{\alpha+1}(n) \equiv f_\alpha^n(n)$. Metzler's videos on YouTube part 4-8 show how these recursively chained functions expands into mind-blowingly large numbers.



$f_\omega(n) > 2 \uparrow^{n-1} n$
$f_{\omega+1}(n) > 2 \to n \to n-1 \to 2$
$f_{\omega 2+k}(n) > n \to n \to n \to k$
$f_{\omega^2}(n) > n \to n \to \cdots \to n \to n$
$\vdots$ (chain with n arrows)

A sequence of ordinal indexed functions with increasing growth is called a **fast-growing hierarchy**.

Even with f_{ω^2} we have still not reached the Conway notation if we allow towers of arrow-counting. To go beyond this let's look at an example:
$f_{\omega^4}(7) = f_{\lim_n \omega^3 n}(7) = f_{\omega^3 7}(7) = f_{\omega^3 6+\omega^3}(7) = f_{\omega^3 6+\omega^2 7}(7) = \cdots$
$= f_{\omega^3 6+\omega^2 6+\omega 6+7}(7) = f^7_{\omega^3 6+\omega^2 6+\omega 6+6}(7).$
This descent in limit ordinals until a successor ordinal is reached looks much like how numbers are written in a positional system with base ω. Every ordinal can be written in a unique way in **Cantor normal form**:

$\alpha = \omega^{\beta_1}c_1 + \omega^{\beta_2}c_2 + \cdots + \omega^{\beta_k}c_k$ with $k, c_1, c_2, \ldots, c_k \in \mathbb{Z}^+$ and ordinals $\beta_1 > \beta_2 > \cdots > \beta_k \geq 0$

β_1 is called the degree of α with $\beta_1 \leq \alpha$. The base can be any ordinal δ>1 (finite, successor or limit ordinal) and c_i will be positive ordinals smaller than δ. ε_0 is a unique ordinal, the first ordinal with degree not less than itself, $\varepsilon_0 = \omega^{\varepsilon_0} \rightarrow \deg(\varepsilon_0) = \varepsilon_0$.

$f_{\omega^\omega}(3) = f_{\omega^3}(3) = f_{\omega^2 3}(3) = f_{\omega^2 2+\omega^2}(3) = f_{\omega^2 2+\omega 3}(3) = f_{\omega^2 2+\omega 2+3}(3) =$
$f^2_{\alpha_0} f^2_{\alpha_0-1} f^2_{\alpha_0-2} f^2_{\alpha_{-1}} f^2_{\alpha_{-1}-1} f^2_{\alpha_{-1}-2} \cdots f^2_{\alpha_{-5}} f^2_{\alpha_{-5}-1} f^3_{\alpha_{-5}-2}(3)$

$f^3_{\alpha_{-5}-2}(3) = f^2_{\omega^2}\big(f_{\omega^2}(3)\big) =$
$f^2_{\omega^2}(3 \rightarrow^3 3) =$
$f_{\omega^2}\big(\, (3 \rightarrow^3 3) \rightarrow^{(3 \rightarrow^3 3)} (3 \rightarrow^3 3)\,\big) = M$
M is a Conway tower with 4 levels.
$f^2_{\alpha_{-5}-1}(M) = f_{\omega^2+1}(f^M_{\omega^2}(M))$
Way beyond a Conway tower of M levels and that is just the beginning of a long chain of functions to unpack, and then there are ever larger numbers like $f_{\varepsilon_{\varepsilon_0}}(3)$.

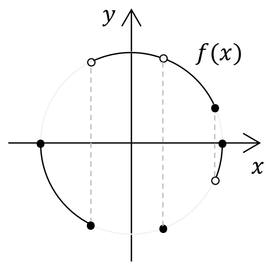
> Alternating between diagonalizing for limit ordinals and downsizing with successor ordinals at:
> $\alpha_0 = \omega^2 2 + \omega 2 + 2$
> $\alpha_{-1} = \omega^2 2 + \omega + 2$
> $\alpha_{-2} = \omega^2 2 + 2$
> $\alpha_{-3} = \omega^2 + \omega 2 + 2$
> $\alpha_{-4} = \omega^2 + \omega + 2$
> $\alpha_{-5} = \omega^2 + 2$

Ordinal numbers are important for proof theory and the fundamentals of mathematics. As an appetizer for things to come, some properties of ε_0, a number with deep links to Peano arithmetic that is based on recursion and proof by induction. Gödel proved that any mathematical system strong enough to contain Peano arithmetic has true statements that are not provable within the system. Diagonalization is essential in the proof of **Gödel's incompleteness theorem**. No proof within Peano arithmetic can show that an algorithm to compute f_{ε_0} will ever stop. Slower functions like $f_{\omega\uparrow\uparrow 100}$ can be handled. Are there any "natural" problem where a function with a growth rate like f_{ε_0} occurs? Yes, there is.

3.12 Differential equations

Equations containing functions are different from equations of variables. If $x^2 + y^2 = r^2$ is viewed an equation of variables $(x, y) \in \mathbb{R}^2$ and a constant $r \in \mathbb{R}$ then the solutions form a circle of radius r. Functions have domains and any solution $f \in \mathcal{F}(D_f \subseteq \mathbb{R}, \mathbb{R})$ to the corresponding functional equation $x^2 + f(x)^2 = r^2$ must have $D_f \subseteq [-r, r]$ but every restriction of the domain will give another function $f|_{A \subset D_f}$ that is also a solution.

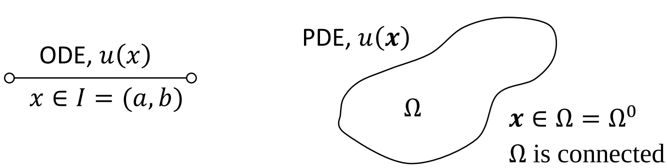
Functions are by definition single-valued but nothing prevents us from freely constructing f out of parts from both the upper and lower semi-circle. Any partition of $D_f = A \cup B$ with $A: f(x) = \sqrt{r^2 - x^2}$ and $B: f(x) = -\sqrt{r^2 - x^2}$ will do if continuity is not a requirement.



Differential equations are functional equations that contain a derivative of a function that is supposed to solve the equation. Since the derivative at a point requires a neighborhood the domain of a solution is assumed to be open. To avoid piecewise constructed solutions in domains of disconnected open pieces it is assumed that the domain is an open interval I of maximal extent. $y' = 1/x$ splits into two separate problems, one for $I = (0, \infty)$ and one for $I = (-\infty, 0)$. Both intervals are solved by $y = \ln|x| + C$. In a more general setting the domain of the solution should be a non-empty connected open set of maximal extent. The term "domain" is also used in a set-context to specify this kind of set in any space where openness and connectivity applies.

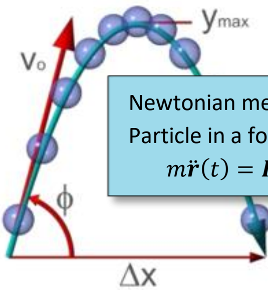
Definition.
Domain is a non-empty connected set in a topological space.

Differential equations are divided into ordinary (ODE) when there is one variable $u(x)$ and partial (PDE) when there are more than one $u(x_1, \ldots, x_n)$. PDEs have equations containing partial derivatives $\partial^{|\alpha|} u / \partial x^\alpha$ and ODEs have ordinary derivatives $d^n u / dx^n$. The domain of u for an ODE will be an interval $I = (a, b)$ with $a < b$ and $a, b \in \mathbb{R} \cup (-\infty, \infty)$ and the domain of u for a PDE will be a set Ω that is a domain of $\mathbb{R}^n$.



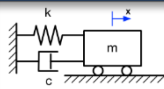
Differential equations
Ordinary differential equations (ODE) $F\left(x, y, y', \dots, y^{(n)}\right) = 0$ Function (y) has one independent variable (x). Order of the equation is n.
Linear $a_n(x)y^{(n)} + \cdots + a_1(x)y'(x) + a_0(x)y(x) + r(x) = 0$ $a_i(x)$ and $r(x)$ are continuous in x. $r(x)$ is the source term.
Homogeneous, if $r(x) = 0$.
Nonhomogeneous, if $r(x) \neq 0$.
Nonlinear F cannot be written in linear form as above.
Autonomous F has no dependence on x, $\boldsymbol{F}\left(\boldsymbol{y}, \boldsymbol{y}', \dots, \boldsymbol{y}^{(n)}\right) = 0$ for a system of ODEs.
Explicit form, equation is in form of $y^{(n)} = G\left(x, y, y', \dots, y^{(n-1)}\right)$.
Implicit form, equation is not given in explicit form.
System of ODEs $\boldsymbol{F}\left(x, \boldsymbol{y}, \boldsymbol{y}', \dots, \boldsymbol{y}^{(n)}\right) = 0$ System of coupled equations with $\boldsymbol{F}$ and $\boldsymbol{y}^{(k)}$ vector valued functions. In explicit form: $y_i^{(n)} = F_i\left(x, \boldsymbol{y}, \boldsymbol{y}', \dots, \boldsymbol{y}^{(n)}\right), \ i = 1, \dots, m$ An ODE of order >1 is usually rewritten as a system of ODE of order = 1.
Partial differential equations (PDE) $F\left(x_i, u, \partial^{\lvert\alpha\rvert} u / \partial x^{\alpha}\right) = 0$ Function $u(x_1, \dots, x_n)$ has $n > 1$ variables. Order of equation is the largest $\lvert\alpha\rvert$ among the arguments of F. Linear PDEs of 2nd order of two variables x and y: $Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$ where $A, \dots, G$ may depend on x and y. In regions where the discriminant: $B^2 - 4AC < 0$, the equation is **elliptic**. $B^2 - 4AC = 0$, the equation is **parabolic**. $B^2 - 4AC > 0$, the equation is **hyperbolic**.
Linear An equation with differential operator form $L[u] = f$ where the differential operator L is linear in u and all its derivatives. When $f = 0$ it is homogeneous and solutions form a vector space.
Nonlinear A differential equation that is not linear.

Differential equations are very common in applications to model reality, especially in physics. Ordinary differential equations often used to describe time dependence in a system. Dot notation is common for time derivatives, displacement $\boldsymbol{s}(t)$ → velocity $\boldsymbol{v}(t)$=$\dot{\boldsymbol{s}}(t)$ → acceleration $\boldsymbol{a}(t)$=$\dot{\boldsymbol{v}}(t)$=$\ddot{\boldsymbol{s}}(t)$.

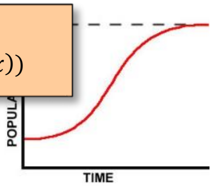


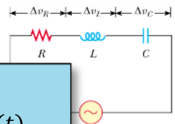
Newtonian mechanics
Particle in a force field
$m\ddot{\boldsymbol{r}}(t) = \boldsymbol{F}(\boldsymbol{r}, t)$

Oscillator
Mass m, Damping c
Spring constant k
$m\ddot{x} + c\dot{x} + kx = 0$



Population growth
$f'(x) = f(x)(1 - f(x))$

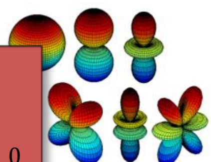




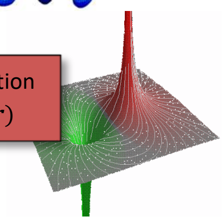
Electronics
RLC-circuit, source $u(t)$
$L\ddot{q} + R\dot{q} + q/C = u$

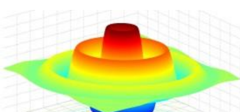
(blue)	ODE linear homogeneous
(light blue)	ODE linear nonhomogeneous
(orange)	ODE nonlinear
(red)	PDE linear elliptic
(pink)	PDE linear parabolic
(purple)	PDE linear hyperbolic
(yellow)	System of nonlinear PDEs

Laplace's equation
$\Delta u(x, y, z) = 0$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

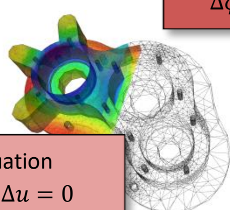


Poisson's equation
$\Delta\varphi(\boldsymbol{r}) = f(\boldsymbol{r})$

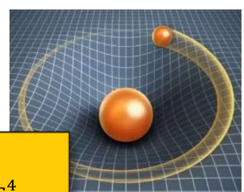




Wave equation
$\partial^2 u/\partial t^2 - c^2 \Delta u = 0$

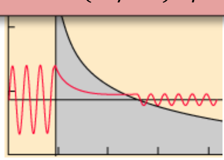


Heat equation
$\partial u/\partial t - \alpha\Delta u = 0$



Einstein Field Equations
$R_{\mu\nu} - R g_{\mu\nu}/2 = 8\pi G T_{\mu\nu}/c^4$
Source of gravity $T_{\mu\nu}$
$R_{\mu\nu} = F^1_{\mu\nu}(\partial g^{|\alpha|}_{ij}/\partial x^\alpha)$
$R = F^2_{\mu\nu}(\partial g^{|\alpha|}_{ij}/\partial x^\alpha)$
$|\alpha| = 0,1,2$

Schrödinger equation
for $\psi(\boldsymbol{r}, t)$ in potential $V(\boldsymbol{r}, t)$
$i\hbar\, \partial\psi/\partial t = -(\hbar^2/2m)\Delta\psi + V\psi$



The general solution of an ODE will contain constants, as many as the order of the equation:

$$\frac{du}{dx} = 0 \Leftrightarrow u(x) = C \quad \frac{d^2u}{dx^2} = f(x) \Leftrightarrow u(x) = \int_{x_0}^{x} f(x)dx + C_1 x + C_2$$

A unique solution of an n-th order ODE requires n constraints:
A: $u^{(i)}(x_0) = y_i$ B: $u(x_i) = y_i$ $(i = 0, \dots, n-1)$ or a mix of A and B.
Type A is called an initial value problem, even though x_0 can be any $x \in I$.
Type B becomes a boundary value problem when $n = 2$, $x_0 = a$ and $x_1 = b$.

Initial value problems with time as a variable describe deterministic systems starting from a given state at an initial time t_0.

For the corresponding PDE: $\partial_x u(x,y) = 0 \Leftrightarrow u(x,y) = f(y)$. The solution of PDEs contain arbitrary functions. If $\Omega = \{(x,y)|x > 0\}$ is the domain of u then $\partial\Omega = \{(0,y)|y \in \mathbb{R}\}$ and $u|_{\partial\Omega} = f_0(y)$ would fix a unique solution $u(x,y) = f_0(y)$ but it is not always true that boundary values will guarantee existence or uniqueness for a solution to a PDE.

How do you solve a differential equation? One type we have already done $y'(x) = f(x), x \in I$. When $f \in C^0(I)$ the fundamental theorem of calculus gives a solution $y_0(x) = \int_{x_0}^{x} f(x)\,dx$. If f is merely Riemann integrable like $f(x) = [x \geq 0] \cdot 1$ then the integral will usually lack derivative somewhere.
If $F(x)$ is any solution then $F'(x) = y_0'(x)$ and by the mean value theorem:
$H(x) \equiv F(x) - y_0(x) \rightarrow H(x) - H(x_0) = (x - x_0)H'(\xi)$ with $\xi \in (x_0, x)$.
$H'(\xi) = 0 \rightarrow H(x) = H(x_0) \rightarrow F(x) = y_0 + C$ for some $C \in \mathbb{R}$.
$y'(x) = f(x)$ with $y(\alpha) = \beta$ has a unique solution $y(x) = \int_{\alpha}^{x} f(x)dx + \beta$.

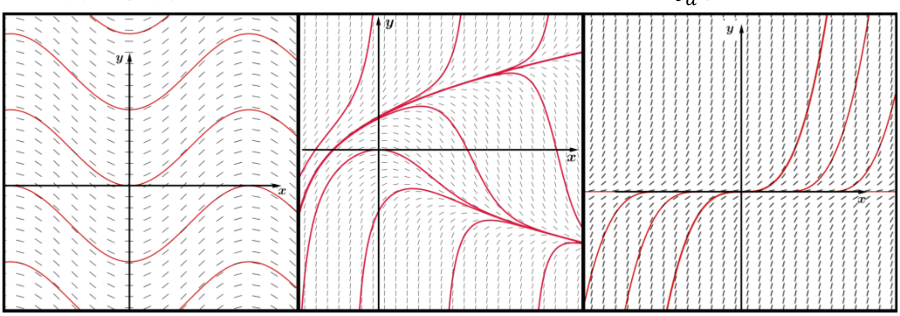


Fig. 3.12.1 Slope fields and solutions for $y' = \sin(x)$, $y' = y^2 - x$ and $y' = 3y^{2/3}$.

Slope fields of $y' = f(x,y)$ with f continuous in a domain $\Omega \subseteq \mathbb{R}^2$ suggests that through every $(x_0, y_0) \in \Omega$ will pass a solution that could be extended in both directions towards the boundary $\partial\Omega$. The last example shows uniqueness is not guaranteed. $y' = 3y^{2/3}$ has several solutions going through (0,0), such as $y(x) = x^3$ and $y(x) = [x \leq -1] \cdot (x+1)^3 + [x \geq 1] \cdot (x-1)^3$.

There is a set of theorems on existence and uniqueness of solutions to ODE’s given in explicit form. With t as independent variable they are as follows.

Theorem. (Peano existence theorem)
Let D be an open subset of $\mathbb{R} \times \mathbb{R}$ and $f\colon D \to \mathbb{R}$ a continuous function on D, then the following initial value problem with $(t_0, y_0) \in D$

$$y'(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

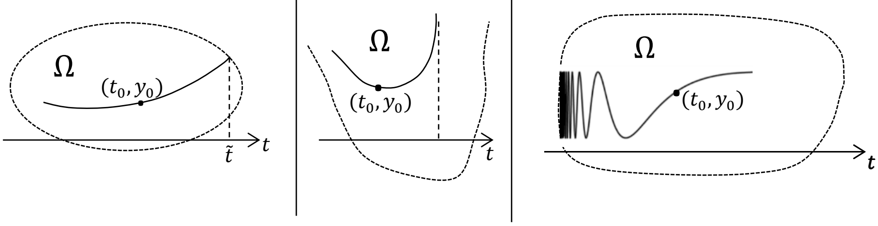
has a local solution $y\colon I \to \mathbb{R}$ in an open interval $I \ni t_0$.

Uniqueness of solutions requires more than continuity of f. It can be attained with Lipschitz continuity which is explained in appendix C together with proofs of the theorems on this page.

Theorem. (Picard-Lindelöf theorem, existence and uniqueness of solution)
For the same initial value problem as above with f satisfying a Lipschitz condition in a neighborhood of (t_0, y_0) the local solution will be unique.

Theorem. (Global version of existence and uniqueness)
If f is continuous in a domain Ω and satisfies a Lipschitz condition in a neighborhood of every point $(t, y) \in \Omega$ then there is a maximal extension of the solution through (t_0, y_0) to an open interval I s.t. one of the following three conditions holds when t approach an edge of I, which can be $\pm\infty$.

$(t, y(t)) \to (\tilde{t}, \tilde{y}) \in \partial\Omega$ | $|t| + |y(t)| \to \infty$ | $(t, y(t)) \to \partial\Omega$, no limit point.



The theorems are true also for higher orders and systems of ODE’s. Existence and uniqueness give us an alternative way to introduce and define functions. $y = e^x$ is the unique solution to $y' = y$ with $y(0) = 1$. $\sin x$ and $\cos x$ are solutions to $y'' = -y$ with different inital values for $y(0)$ and $y'(0)$. This can be rewritten as a first order ODE for $\boldsymbol{y} = (y_1, y_2)$ where $y_2 = {y_1}'$.

$$y = e^x \Leftrightarrow \begin{cases} y' = y \\ y(0) = 1 \end{cases} \quad \boldsymbol{y} = (\sin x, \cos x) \Leftrightarrow \begin{cases} \boldsymbol{y}' = (y_2, -y_1) \\ \boldsymbol{y}(0) = (0,1) \end{cases}$$

A polygonal chain $(t_n, \boldsymbol{y}_n)$ based on the slope field with $t_n = t_0 + nh$ and $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ gives an approximation of the graph $(t, \boldsymbol{y}(t))$ with increasing precision with decreasing step length h. This is Euler’s method.

Runge-Kutta methods

Euler's method is one in a series of methods to find approximate solutions to $\dot{y} = f(t, y), y(t_0) = 0$. A better method is Runge-Kutta method, RK4.

$$y_{n+1} = y_n + h \cdot \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} \qquad k_1 = f(t_n, y_n)$$
$$t_{n+1} = t_n + h \qquad k_2 = f(t_n + h/2, y_n + hk_1/2)$$
$$k_3 = f(t_n + h/2, y_n + hk_2/2)$$
$$k_4 = f(t_n + h, y_n + hk_3)$$

With f independent of y the equation corresponds to integration

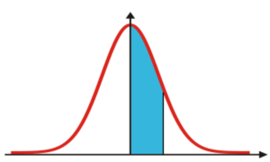
$$y'(t) = f(t) \to y(t) = \int_{t_0}^{t} f(s)ds + y_0$$

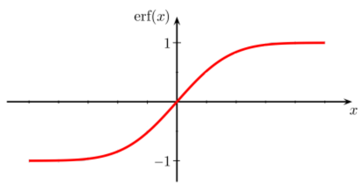
Runge-Kutta's method becomes equal to Simpson's rules for calculating integrals. Simpson's method is based on adjusting quadratic polynomials in each interval of a partitioning with constant step length.

Numerical methods are fine but they do not replace solutions expressed in terms of **elementary functions**. This is a well-defined group based on finite compositions with arithmetic operators $(+, -, \times, \div)$, constants, exponentials, logarithms and solutions to algebraic equations. Trigonometric functions are elementary, $\sin z = (e^{iz} - e^{-iz})/2$ and $\arcsin z = -i \ln(iz + \sqrt{1 - z^2})$.

Elementary functions are closed under derivation but not under integration. The error function, the integral of the normal distribution, very common in probability, statistics and physics is not an elementary function.

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad \mathrm{D}^{-1}(e^{-x^2}) \propto \operatorname{erf}(x)$$

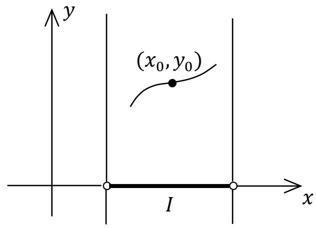




Other functions with antiderivatives that are not elementary functions are $x^x, \sqrt{1 - x^4}, 1/\ln x, e^x/x, \sin x^2$ and $\sin(x)/x$. These results were given by Liouville. Extending the definition of elementary functions to include antiderivation gives a broader class, the Liouvillian functions. They are all solutions to algebraic differential equations but the opposite is not true. Bessel functions is an example. They solve $x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$ and they are not Liouvillian functions.

Examples of differential equations that can be solved in "Liouvillian form" are **linear ODE of first order**: $k(x)y'(x) + g(x)y(x) = h(x)$.
Divide by $k(x)$ for an interval where $k(x) \neq 0$.
$y' + gy = h$ for some $g, h \in C^0(I)$.
With $G(x) = \int_{x_0}^{x} g(t)dt$:
$e^G(y' + gy) = e^G h$ (e^G integrating factor)
$D(e^G y) = e^G h$

$$y(x) = e^{-G(x)}\left(\int_{x_0}^{x} e^{G(t)}h(t)dt + C\right)$$

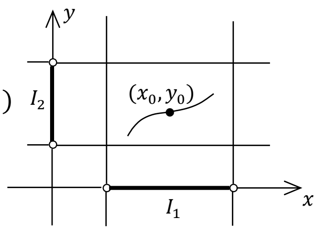


Another "solvable" example is when the equation has **separable variables.**
$g(y)y'(x) = h(x)$ where $g \in C^0(I_2)$ and $h \in C^0(I_1)$. ($g(y)dy = h(x)dx$)
With $G(y) = \int_{y_0}^{y} g(t)dt$ and $H(x) = \int_{x_0}^{x} h(s)ds$

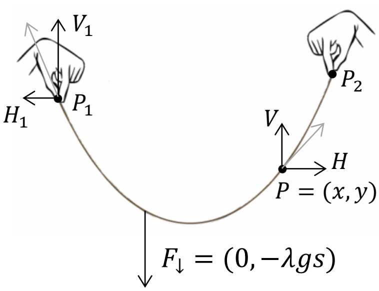
$$\frac{d}{dx}(G(y(x)) = \frac{d}{dx}H(x) \Leftrightarrow G(y(x)) = H(x) + C$$

Implicit form for the solution passing through (x_0, y_0)

$$\int_{y_0}^{y} g(t)dt = \int_{x_0}^{x} h(s)ds$$



Example (Chain curve a.k.a catenary)
A string hangs from two points P_1 and P_2.



s: Arc length P_1P
λ: Linear density of string
g: Graviational field (N/kg)
H_1, V_1, H, V: Horizontal and vertical force components at P_1 and P.
Static equilibrium $\rightarrow \begin{cases} H + H_1 = 0 \\ V + V_1 = \lambda gs \end{cases}$

Tension in the string gives the force a tangential direction.

$$\frac{dy}{dx} = \frac{V}{H} = \frac{\lambda gs - V_1}{-H_1} = \alpha s + \beta \ (\alpha > 0) \rightarrow \frac{d^2y}{dx^2} = \alpha\frac{ds}{dx} = \alpha\sqrt{1 + (dy/dx)^2}$$

This is a 1st order ODE in $z(x) = y'(x)$ with separable variables.

$$\frac{z'}{\sqrt{1+z^2}} = \alpha \rightarrow \text{arsinh}(z) = \alpha(x - C) \rightarrow z = \sinh(\alpha(x - C))$$

$y = D^{-1}(z) = \frac{\cosh(\alpha(x-C))}{\alpha} + D$ Translating $x - C \curvearrowright x$ and $y - D \curvearrowright y$

$y(x) = \dfrac{\cosh(\alpha x)}{\alpha} = \dfrac{e^{\alpha x} + e^{-\alpha x}}{2\alpha}$ The chain curve or catenary

Our last example of equations to look at are linear ODE of order n:

$$\sum_{k=0}^{n} a_k(x)y^k(x) = g(x) \quad y(x) \in \mathrm{C}^n(I) \wedge a_k(x), g(x) \in \mathrm{C}^0(I)$$

x belongs to a domain $I \subseteq \mathbb{R}$ but the range of functions can be either real or complex. The equation is homogeneous if$g(x) \equiv 0$.

A useful concept and notation here is to introduce a differential operator $\mathcal{L}(y)$ which is a function that operates on a function to produce another function, $\mathcal{L}: \mathrm{C}^n(I) \to \mathrm{C}^0(I)$, $y \curvearrowright \mathcal{L}y = \sum_{k=0}^{n} a_k D^k y$.

Solutions of $\mathcal{L}y_h$=0 form a linear space $\mathcal{L}y_1$=0, $\mathcal{L}y_2$=0 $\Rightarrow \mathcal{L}(c_1y_1 + c_2y_2)$=0. Adding any of the solutions of $\mathcal{L}y_p$=g called a **particular solution** gives the solution space for $\mathcal{L}y$=g. When the coefficients a_k are constants the solutions form an n-dimensional affine space, a linear space with no particular point of origin, $y = y_p + y_h = y_p + \sum_{k=1}^{n} c_k y_k$ with $c_k \in \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{L}y_k$=0.

Homogeneous linear ODE with constant coefficients of order n, $\mathcal{L}y = 0$:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0 = 0$$

$\mathcal{L}(e^{rx}) = l(r)e^{rx}$ Where $l(r) \equiv r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$ is the **characteristic polynomial.** $l(r_0) = 0 \Longrightarrow y = e^{r_0 x}$ solves $\mathcal{L}y = 0$.

Theorem.
Let $l(r) = \prod_{k=1}^{\nu}(r - r_k)^{n_k}$ be the characteristic polynomial of $\mathcal{L}y = 0$ then $y(x) = \sum_{k=1}^{\nu} P_k(x)e^{r_k x}$ with polynomials $P_k$of $\deg(P_k)$<n_k solves $\mathcal{L}y = 0$ and in reverse, all solutions to $\mathcal{L}y = 0$ are of this form.

Real solutions when $a_k \in \mathbb{R}$ arise from conjugate roots:
$\mathfrak{Re}\left(c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}\right) = e^{\alpha x}(C\cos\beta x + D\sin\beta x) = Ae^{\alpha x}\sin(\beta(x+\delta))$

Proof.
I.
y linear combination of functions $x^p e^{r_k x}$ with $p < n_k \Rightarrow \mathcal{L}y = 0$.
$D\left(e^{rx}z(x)\right) = e^{rx}(D + r)z(x) \to$ induction $\to D^k\left(e^{rx}z(x)\right) = e^{rx}(D + r)^k z(x)$

$$l(r) = \prod_k (r - r_k)^{n_k} \to l(D + r_i) = l_i(D)D^{n_i} \begin{cases} \text{where } r_i \text{ is a root of } l(r) \\ \text{and } l_i \text{ is a polynomial} \\ \text{of degree } n - n_i \end{cases}$$

$$\mathcal{L}(x^p e^{r_i x}) = l(D)(e^{r_i x}x^p) = e^{r_i x} l(D + r_i)x^p = e^{r_i x} l_i(D) \underbrace{D^{n_i} x^p}_{p<n_i} = 0$$

Linearity of $\mathcal{L} \Rightarrow \mathcal{L}y = 0$
II.
$\mathcal{L}y = 0 \Rightarrow y$ linear combination of functions $x^p e^{r_k x}$ with $p < n_k$,
is left as an exercise for the reader.

Now we only need to find one solution y_p to $\mathcal{L}(y) = g$ to find them all since if $y_{p'}$ where any other solution then $y_{p'} = y_p + (y_{p'} - y_p)$ and $y_{p'} - y_p$ belongs to the solution set of $\mathcal{L}(y) = 0$ that we already know.

$\mathcal{L}(y) = g$ has already been solved in the first order: $y' - ry = g$ by using an integrating factor, $y(x) = e^{rx}\int_{x_0}^{x} e^{-rt}g(t)dt = \int_{x_0}^{x} e^{r(x-t)}g(t)dt$. Applying a solution of the form $y(x) = \int_{x_0}^{x} K(x-t)g(t)$ for order n leads to:

Theorem.

$$\begin{cases}\mathcal{L}(K) = 0\\ K^{(i)} = 0 \text{ for } i = 0,1,\dots,n-2\\ K^{(n-1)}(0) = 1\end{cases} \Rightarrow \quad \begin{array}{c} y(x) = \displaystyle\int_{x_0}^{x} K(x-t)g(t)dt \\ \text{solves } \mathcal{L}(y) = g\end{array}$$

For some $g(x)$ solutions to $\mathcal{L}(y) = g$ can be found by making an ansatz.

I $g(x) = P_n(x)$ a polynomial of degree n, then there is a solution with $y_p(x) = x^m Q_n(x)$ with Q a polynomial of degree n, m the multiplicity of $r = 0$. (non-roots have multiplicity zero)

II $g(x) = P_n(x)e^{kx}, k \in \mathbb{C}$ reduces to case I. with ansatz $y_p(x) = e^{kx}z(x)$.

III If $a_k \in \mathbb{R}$ then $\mathcal{L}(y_p) = g(x) \Rightarrow \begin{cases}\mathcal{L}(Re(y_p)) = Re(g(x))\\ \mathcal{L}(Im(y_p)) = Im(g(x))\end{cases}$

$g(x) = P_n(x)e^{\alpha x}\cdot\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$ are treated by handling $P_n(x)e^{(\alpha+i\beta)x}$.

IV $\mathcal{L}(y) = g_1 + g_2$ is solved by $y_1 + y_2$ if $\mathcal{L}(y_1) = g_1$ and $\mathcal{L}(y_2) = g_2$.

Examples from physics

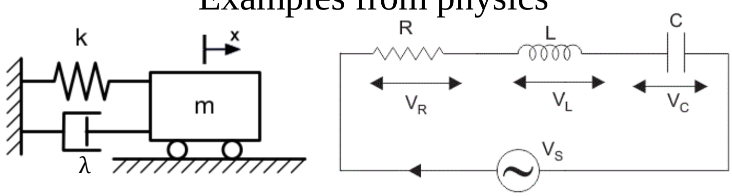


Fig 3.12.2 Damped harmonic oscillator and RLC-circuit.

The harmonic oscillator and the RLC-circuit are physical models that use 2nd order linear ODEs with constant coefficient to describe phenomena and applications of great importance. The oscillator describes motion around an equilibrium position and the RLC-circuit has the power to generate and receive electromagnetic waves, the basis for communication via radio waves.

The harmonic oscillator has a restoring force, modelled by a spring. To first approximation the force is proportional to the displacement. This covers small oscillation for any restoring force.

$F = -k \cdot x$ k: spring constant

$x(t)$: displacement from equilibrium

$m\ddot{x} + kx = 0$ By Newton's 2nd law, $F_{total} = m\ddot{x}$

$r^2 + k/m = 0$

$r_{1,2} = \pm i\sqrt{k/m} \rightarrow x(t) = A\cos(\omega_0 t + \varphi), \;\; \omega_0 = \sqrt{k/m}$

Undamped angular frequency $\omega_0 = 2\pi f$, resonance frequency f.

Most real oscillators have friction, modeled (as a first approximation) by a damping force proportional to velocity.

$F = -\lambda\dot{x}$ λ: damping coefficient

$m\ddot{x} + \lambda\dot{x} + kx = 0 \rightarrow$ 3 types of damping depending on if the roots are real, imaginary or a double root.

$\zeta = \lambda/(2\sqrt{mk})$: damping ratio

With a driving force $F(t) = F_0\cos\omega t$ acting on the oscillating object:

$m\ddot{x} + \lambda\dot{x} + kx = F_0\cos\omega t$

$x(t) = x_p(t) + x_h(t)$ $x_h(t)$ is transient and is damped away

$x_p(t)$ is steady state motion

$$x_p(t) = \frac{F_0}{m\omega Z}\sin(\omega t + \varphi)$$

$Z = \sqrt{(2\omega_0\zeta)^2 + (\omega_0^2 - \omega^2)/\omega^2}$: mechanical impedance

A solution with no damping, $\zeta = 0$:

$$x(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t) = \frac{-2F_0}{m(\omega_0^2-\omega^2)}\sin\frac{\omega-\omega_0}{2}t\sin\frac{\omega+\omega_0}{2}t$$

has an oscillating amplitude of maximal size $2F_0/(m|\omega_0^2 - \omega^2|)$.

A driving force like the wind can cause a bridge to collapse if it creates a force with frequency close to the resonance frequency of the bridge.

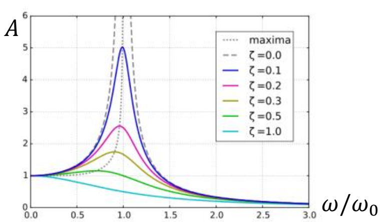
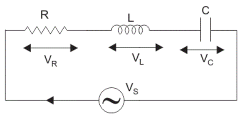
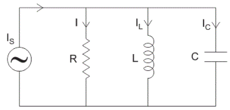


Fig. 3.12.3 Amplitudes for different dampings, Tacoma bridge collapse in 1940.



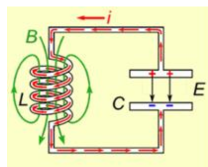


$$V_R = RI = R\dot{Q}$$
$$V_L = -L\dot{I}$$
$$V_C = -Q/C$$

Kirchoff's law gives $RI = V_L + V_C + V_S \rightarrow L\ddot{I} + R\dot{I} + I/C = \dot{V}_S$.
The parallell circuit follows a similar equation, a dual version with rearranged parameters, voltage to solve for and a current as source term.

Spring	Serial circuit	Parallell circuit
displacement $x(t)$	charge $q(t) = \int_t Ids$	$\varphi(t) = \int_t Vds$
velocity $v = \dot{x}$	current $I = \dot{q}$	voltage $V = \dot{\varphi}$
mass m	inductance L	capacitance C
spring constant k	$1/C$	$1/L$
damping λ	resistance R	$1/R$
Applied force $F(t)$	voltage source $V_S(t)$	current source $I_S(t)$
$\omega_0 = \sqrt{k/m}$	$\omega_0 = 1/\sqrt{LC}$	$\omega_0 = 1/\sqrt{LC}$
$m\ddot{x} + \lambda\dot{x} + kx = F$	$L\ddot{I} + R\dot{I} + I/C = \dot{V}_S$	$C\ddot{V} + \dot{V}/R + V/L = \dot{I}_S$

A pure LC-circuit generates an undamped harmonic oscillation between a magnetic dipole field from the coil and an electric dipole field from the capacitor.



Forced oscillations with $V = V_0 \sin \omega t$

$$\ddot{I} + R/L\,\dot{I} + 1/C\,I = \omega V_0 \cos \omega t$$

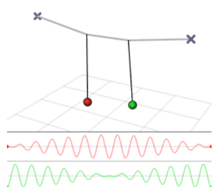
$$r_{1,2} = -\gamma \pm i\omega_h \rightarrow I_h \propto e^{-\gamma t}\sin(\omega_h t + \alpha)$$

Reactance $R = \omega L - 1/(\omega C)$

Impedance $Z = \sqrt{R^2 + X^2}$ $, I_0 = \frac{V_0}{Z} \rightarrow I_p = I_0 \sin(\omega t - \varphi)$

Two separate RLC-circuits with electromagnetic induction from one circuit to the other leads to two coupled differential equations. The same works for mechanically coupled oscillators.

$$\begin{cases} L_1\ddot{I}_1 + R_1\dot{I}_1 + I_1/C_1 = -M\ddot{I}_2 \\ L_2\ddot{I}_2 + R_2\dot{I}_2 + I_2/C_2 = -M\ddot{I}_1 \end{cases}$$



The discovery of electrical oscillations where made by F. Savay in 1826 with Leyden jars as capacitors and with a wire around an iron needle as inductor. In 1853 Kelvin calculated and demonstrated the resonance frequency of an RLC-circuit. Work by Maxwell and Hertz lead the discovery that radio waves could be generated from one circuit to be picked up by another circuit. The first radio system was built in 1900 by Guglielmo Marconi.

To each differential equation there is a discretized version, like the Euler method $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{f}(t_n, \boldsymbol{y}_n)$ is a discrete version of $\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t))$. A **recurrence relation** defines a sequence like a differential equation defines a function. If the differential equation has a solution in closed form it seems there should be a solution in closed form for the recurrence relation as well.

A homogeneous linear recurrence relation with constant coefficients is:

$$y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots + a_n y_{k-n} \quad k \in \{n, n+1, \ldots\} \quad (*)$$

With $a_n \neq 0$ this is an equation of order n. It has a unique solution for given initial values of $y_0, y_1, \ldots, y_{n-1}$. The term **difference equation** is sometime used as a synonym even though it seems more natural to reserve the term for equations expressed in terms of difference operators

$$\Delta y_k = y_{k+1} - y_k \quad \Delta^2 y_k = y_{k+2} - 2y_{k+1} + y_k \ldots \quad y_{k+n} = \sum_{i=0}^{n} \binom{n}{i} \Delta^i y_k$$

Order 1: $y_k = a_1 y_{k-1} \rightarrow y_k = y_0 a_1^k$

Making an ansatz $y_k = cr^k$ gives a solution to $(*)$ if r is a root of the **characteristic polynomial** of $(*)$:$P(t) = t^n - a_1 t^{n-1} - \cdots - a_{n-1} t - a_n$.

Theorem.
If the characteristic polynomial of $y_k = a_1 y_{k-1} + a_2 y_{k-2} + \cdots + a_n y_{k-n}$ is $P(t) = \prod_{i=1}^{\nu} (t - r_i)^{n_i}$ then the general solution for y_k is given by:

$$y_k = \sum_{i=1}^{\nu} Q_i(k) r_i{}^k \text{ with } Q_i(t) = c_0 + c_1 t + \cdots + c_{n_i - 1} t^{n_i - 1}$$

The most famous example is the Fibonacci sequence $y_k = y_{k-1} + y_{k-2}$ with $y_0 = 0$ and $y_1 = 1$ which becomes $0,1,1,2,3,5,8,13,21,34, \ldots$. The sequence was presented by Fibonacci in his book *Liber Abaci* from 1202. Fibonaci was not the first to study the sequence. Indian mathematicians had used it already in 200 BC to enumerate patterns of long and short syllables (Exercise. 3.10). The Fibonacci numbers occur in so many diverse areas of mathematics that they have their own jounal, *the Fibonacci Quarterly*.

Fibonacci's original application of the sequence was to count rabbits, or the growth potential of any unrestrained population. Start with a young rabbit pair at month 1, $F(0) = 0$ and $F(1) = 1$. Pairs will mate and produce a new pair once a month once they are one month old and they will never die. The number of rabbit pairs in month k will be:

$$F_k = F_{k-1}(\text{aging rabbit pairs}) + F_{k-2}(\text{rabbits old enough to reproduce})$$

$F_n = F_{n-1} + F_{n-2}$

$F_0 = 0$ and $F_1 = 1$

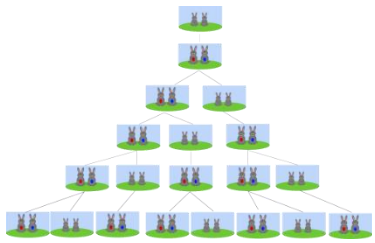
Roots of characteristic polynomial:

$t^2 - t - 1 = 0 \to r_{1,2} = \left(1 \pm \sqrt{5}\right)/2$

Golden ratio $\varphi \equiv \left(1 + \sqrt{5}\right)/2 = 1.618 \ldots$

$\psi \equiv \left(1 - \sqrt{5}\right)/2 = 1 - \varphi = -0.618 \ldots$

$$\begin{matrix} F_0 = 0 \\ F_1 = 1 \end{matrix} \to F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \to \lim_{n \to \infty} \frac{F_{n+k}}{F_n} = \varphi^k$$



Exponential growth

No biotope consists of a single species and animals die. A model that takes this into account is the predator-prey model. It tracks the number of predators and preys, like rabbits and foxes. The continuous version of this model is the **Lotka-Volterra equations**, two coupled differential equations. A drawback of using a continuous model is that it does not capture the possibility of a species going extinct when its numbers are low. This is the atto-fox problem, with only 10^{-18} foxes left they could still recover but the purpose of using a model is to simplify things and still capture something essential of what is modelled. Some aspects are bound to be lost in a model.

$$\begin{cases} \dot{x} = \alpha x - \beta x y \\ \dot{y} = \delta x y - \gamma y \end{cases} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}^+$$

$x(t)$ Number of prey

$y(t)$ Number of predators

$\dot{x}, \dot{y}$ Growth rates of prey and predators

α Rate of growth of prey without predation

β Rate of predation, xy measures likelihood of contact predator-prey

δ Growth rate of predators, proportional to number of preditors and pray

γ Loss rate due to competition among predators for prey

Among the many underlying assumptions are unlimited food supply for prey. The system is non-linear and has no solution given by elementary functions. Nevertheless, solutions are periodic in t. This can be understood by looking at $\dot{x}/\dot{y} = dx/dy$ and $\dot{y}/\dot{x} = dy/dx = (\delta x y - \gamma y)/(\alpha x - \beta x y) = f(x, y)$, a slope field that turns out to have closed orbits in **phase space** (x, y).