

Tricky problems with %, ‰ and ppm

1.
A cup contains a mixture of two liquids A and B. The share of A in percent (%) equals the share of B in permille (‰). The cup contains one litre. How many litres of A and how many litres of B is there? Give your answer in decimal form.

2.
A mixture of three liquids A, B and C contains as many % of A as ‰ of B as ppm of C. what is the share of A, B and C? Give your answer in decimal form.

Fractions are given in decimal form as follows:

Fractions have either finite or infinite decimal representation.

Examples of finite representation: $1/2 = 0.5$ $7/4 = 1.75$ $17/8 = 2.125$

If the decimal representation has an infinite number of digits it always ends in a sequence of digits that repeats indefinitely. The repeating digit sequence is marked by a dash above the digits, an overline.

Examples:

$$\frac{1}{3} = 0.333 \dots = 0.\overline{3} \quad \frac{1}{99} = 0.010101 \dots = 0.\overline{01} \quad \frac{211}{7} = 30.142857142857 \dots = 30.\overline{142857}$$

Calculating $\frac{4571}{990}$ with long division shows why fractions end the way they do:

$$\begin{array}{r}
 4,617 \\
 990 \overline{)4571,000} \\
 \underline{-3960} \quad (=4 \cdot 990) \\
 6110 \\
 \underline{-5940} \quad (=6 \cdot 990) \\
 1700 \\
 \underline{-990} \quad (=1 \cdot 990) \\
 7100 \\
 \underline{-6930} \quad (=7 \cdot 990) \\
 170
 \end{array}$$

The remainders, marked in grey are always between 0 and the denominator-1. If a remainder is 0 the division is completed and the fraction has a finite number of digits. If the remainder never becomes 0 it must eventually return to a previous remainder since the possible number of remainders are limited to $\{1, \dots, 989\}$. When a remainder returns the digits in the fraction will repeat. The number of repeating digits can never become larger than the denominator - 1.

$$\frac{4571}{990} = 4.6171717 \dots = 4.6\overline{17}$$

Clue problem 1:

If the mixture is 90% liquid A and 90‰ liquid B then there is 0.9 litres of A and 0.09 litres of B, in total 0.99 litres. The shares must therefore be somewhat larger than 90% and 90‰.

Solution problem 1:

$$x\% + x\text{‰} = 1$$

$$\frac{x}{100} + \frac{x}{1000} = 1 \quad (\times 1000)$$

$$10x + x = 1000$$

$$x = \frac{1000}{11} = 90.9090 \dots = 90.\overline{90} \quad A: 0.\overline{90} \text{ liter} \quad B: 0.0\overline{90} \text{ liter}$$

$$0.90909090 \dots$$

$$+ 0.09090909 \dots$$

$$= 0.99999999 \dots = 1 \quad (\text{If this seems wrong, think of } 3 \cdot 0.333 \dots = 3 \cdot \frac{1}{3})$$

Solution problem 2:

$$x\% + x\text{‰} + x \text{ ppm} = 1$$

$$\frac{x}{100} + \frac{x}{1000} + \frac{x}{1\,000\,000} = 1 \quad (\times 1\,000\,000)$$

$$10\,000x + 1000x + x = 1\,000\,000$$

$$x = \frac{1\,000\,000}{11001} = 90.9008271 \dots$$

Calculating the repeating digits of x by hand is far too tedious. It can be done with Mathematica, `RealDigits[1000000/11001][[1, -1]]`. The result shows 576 repeating digits.

$$\frac{x}{100} =$$

0.909008271975274975002272520679938187437505681301699845468593764203
 25424961367148441050813562403417871102627033906008544677756567584765
 02136169439141896191255340423597854740478138351058994636851195345877
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 47159349150077265703117898372875193164257794745932187982910644486864
 83046995727661121716207617489319152804290519043723297882010726297609
 30824470502681574402327061176256703936005817652940641759840014544132
 3516043996000363603308790109990000

A: 0,90900827197 ... 999000090900827 ...
 B: 0,09090082719 ... 099900009090082 ...
 C: 0,00009090082 ... 901099900009090 ...
 A+B+C=0,9999999999 ... 999999999999999 ... = 1

Another variation on the same theme is: $A+B+C=1$ where B is 1% of A and C is 1‰ of B .

Find a digit sequences $xyz \dots$ that makes the following addition true.

$$A: \quad 0,xyz \dots$$

$$B: \quad 0,00xyz \dots$$

$$C: \quad 0,00000xyz \dots$$

$$A+B+C=0,99999999 \dots$$

The one and only solution for $xyz\dots$ is a repeating sequence based on the following digit sequence:

99008920703755408362293442639181790279304165305294006990029801685131830377
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4817279036841219393867387451609390006039544162929079910099900000

The length of the repeating sequence (period) in the decimal representation of $\frac{100000}{101001}$ is 16640 digits.

Extending to four ingredients. B=1% of A , C=1‰ of B and D = 1 ppm of C gives:

$$A = \frac{100.000.000.000}{101.001.000.001} \text{ with period } 36\ 363\ 996$$

An even longer period is achieved with: B = 1% of A , C = 1 ppm of B , D = 1‰ of C.

$$A = \frac{100.000.000.000}{100.101.000.001} \text{ has a period equal to } 25\ 014\ 018\ 913 \text{ more than } 25 \text{ billion digits.}$$

Written on A4-pages with 75 digits per row and 50 rows per page gives

$$\frac{25 \cdot 10^9}{50 \cdot 75} = 6666666.66 \dots \approx 7 \cdot 10^6 \text{ pages.}$$

25 cm per page means a printout would be 1670 km long.